DESIGN OF BOOLEAN LUM SMOOTHERS THROUGH PERMUTATION COLORING CONCEPT

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Abstract

Rank-order based LUM (lower-upper-middle) smoothers distinguishes by wide range of smoothing characteristics given by filter parameter. Thus, for the capability to achieve the best balance between noise suppression and signal details preservation, the LUM smoothers are preferred in smoothing applications. Thanks to threshold decomposition and stacking properties, the LUM smoothers belong to the class of stack filters. This paper is focused to the derivation of minimal positive Boolean function for LUM smoothers through permutation groups and a coloring concept.

Keywords

LUM smoothers, Boolean function, stack filters, permutation theory, coloring

1. Introduction

A number of nonlinear filters belong to the class of stack filters [4,11,15]. The stack filters posses the threshold decomposition and stacking properties. Thus, the stack filtering consists of decomposing an input signal into a set of binary signals, where the uniform filtering operation is performed and consecutive summing up outputs of binary filters.

Recently developed class of rank-order LUM (lower-upper-middle) smoothers [3,5,7-9] is widely used in image and signal smoothing applications, since these filters well suppress impulse noise and preserve details, simultaneously. LUM smoothers can be expressed as stack filters. The efficient and fast searching algorithm for minimal positive Boolean function (PBF) was derived and introduced [6] and more complex ordering of input samples was eliminated.

In this paper, the detailed analyse of searching algorithm is performed. Considering a number of Boolean elements, the permutations [1,2] of input set must be realized and the reduced set of permutations is obtained through a permutation coloring concept that depends on smoothing level of LUM smoothers. Thus, the minterms of minimal PBF are determined by the set of colored permutations. In addition, the number of minimal PBF minterms for each smoothing level done by LUM smoothers can be obtained.

2. Stack filters

The stack filters are nonlinear filters that window operator is based on a positive Boolean function (PBF) [13]. An N-input Boolean function $f_B(.)$ is said to possess the stacking property if

$$f_B(x_1, x_2, ..., x_N) \geq f_B(y_1, y_2, ..., y_N) \qquad (1)$$

when $x_i \geq y_i$ for all i. The necessary and sufficient condition for a Boolean function to posses the stacking property is that it be a PBF [4,12]. Note, that PBF performs the logical AND and OR operations only, i.e. the negation operation is excluded.

Given K-valued input signal $W = \{x_1, x_2, x_N\} \in \{0,1, ..., K\text{-}1\}^N$. The threshold decomposition [12-15] of W amounts to decomposing it to K-1 binary signals $W^1, W^2, ..., W^{K-1}$, where W^m is defined by

$$W^m = \{x_1^m, x_2^m, ..., x_N^m\} \in \{0,1\}^N \qquad (2a)$$

$$x_i^m = \begin{cases} 1 & \text{if } x_i \geq m \\ 0 & \text{otherwise} \end{cases} \qquad (2b)$$

where $i = 1,2,...,N$. Then, W is expressed by

$$W = \sum_{m=1}^{K-1} W^m \qquad (3)$$

The K-valued stack filter $S_f: \{0,1,...,K\text{-}1\}^N \rightarrow \{0,1,...,K\text{-}1\}^{2M+1}$ based on the PBF $f_{PBF}: \{0,1\}^N \rightarrow \{0,1\}$ can be defined as follows [15]

$$S_f(W) = S_f(\sum_{m=1}^{K-1} W^m) = \sum_{m=1}^{K-1} f_{PBF}(W^m) \qquad (4)$$

where N, K and M are the positive integers.

3. LUM smoothers

A subclass of rank-order based LUM filters (Fig. 1) [3,7], LUM smoothers distinguish by wide range of smoothing characteristics. Level of smoothing done by LUM smoother is given by tuning parameter for smoothing. Thus, LUM smoothers can be designed to best balance between noise suppression and signal-details preservation.

In many applications, the some filters, e.g. medians introduce too much smoothing. The blurring introduced may be more objectionable than the original noise. In case of LUM smoothers varying the filter parameter k changes the level of the smoothing from no smoothing (i.e. identity filter for k=1, where $y^*=x^*$) to the maximum amount of smoothing (i.e. median, $k=(N+1)/2$, where N is a window size). Thus, the smoothing function is created by a simply comparing of processed sample x^* to the lower- and upper-order statistics. If x^* lies in a range formed by these order statistics it is not modified. If x^* lies outside this range it is replaced by a sample that lies closer to the median.

The output of LUM smoother is given by

$$y^* = med\{x_{(k)}, x^*, x_{(N-k+1)}\}, \tag{5}$$

where *med* is a median operator that requires ordering (6) and the choice of central sample from ordered set, N is a window size, x^* is a middle sample of input set, $x_{(k)}$ and $x_{(N-k+1)}$ are lower and upper order statistics of the ordered set given by

$$x_{(1)} \le x_{(2)} \le \ldots \le x_{(N)}. \tag{6}$$

The definition by (5) is equivalent to center-weighted median (CWM) that is given by the median over modified set of observations containing multiple processed samples. However, implementation of the LUM smoother as shown in (5) requires fewer operations [3] than that of (7), since fewer elements must be sorted.

The output of CWM is given by

$$y^* = med\left\{W \cup \left\{\underbrace{x^*, x^*, \ldots, x^*}_{w^*-1}\right\}\right\} \tag{7}$$

In (5) $W=\{x_1, x_2, \ldots, x_N\}$ is the input set determined by filter window and w^* is the weight of the central sample and is assumed to be an odd positive integer. The relationship between the parameter w^* in the CWM and the parameter k in the LUM smoother is

$$w^* = N - 2k + 2. \tag{8}$$

Important fact, that the output of LUM smoother is restricted to be a sample of input set W, thus, it will never cause any undershoot and over shoot.

From statistical properties, the very important impulse noise breakdown probability [3] of LUM smoothers is given by (9). The breakdown probability is the probability of outputting an impulse given a certain probability p of impulses appearing in the input (i.e. in the case of 10% noise, the probability p = 0.1). It is clearly, that the breakdown probability for LUM smoother is decreased with increased parameter k. When p is small, a low breakdown probability can be obtained with relatively small k. By the low value of k can be achieved excellent signal-detail preservation.

$$p_b = p\sum_{i=k-1}^{N-1}\binom{N-1}{i}\left(\frac{p}{2}\right)^i\left(1-\frac{p}{2}\right)^{N-i-1} + (2-p)\sum_{i=N-k+1}^{N-1}\binom{N-1}{i}\left(\frac{p}{2}\right)^i\left(1-\frac{p}{2}\right)^{N-i-1} \tag{9}$$

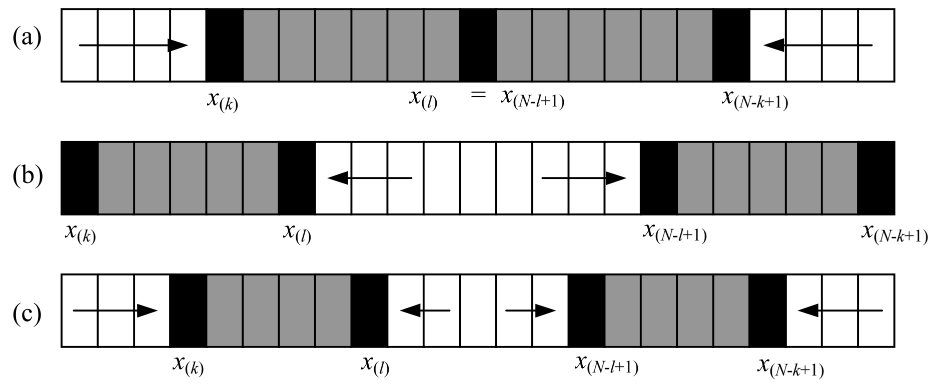

Fig. 1 LUM filter (a) smoother (b) sharpener (c) hybrid smoothing and sharpening filter

4. Boolean representation of LUM smoother

Since the LUM smoothers belong to wide class of rank-order based filters, each smoothing level of LUM smoothers can be expressed by PBF. The search algorithm for minimal PBF of various level of smoothing done by LUM smoothers was developed and introduced [6]. This algorithm simplifies considerably more complicated searching through CWM [15]. In addition, the minimization of PBF is eliminated since the minimal Boolean expression is obtained directly and the design is faster and easier.

Given a tuning parameter k of LUM smoother and a window size N. Then associated input set is given by $(x_1, x_2, ..., x_N)$. The corresponding minimal PBF of LUM smoother can be found by the following procedure:

1. Create the minterms of k elements, each minterm must contain central sample $x_{(N+1)/2}$.
2. Create the minterms of N-k+1 elements without central sample $x_{(N+1)/2}$.

Example 1: Consider k=2 and N=5. The set of minterms corresponding to step 1 is expressed as $\{x_1x_3, x_2x_3, x_3x_4, x_3x_5\}$. Thus, all 2-elements minterms containing central sample x_3 were included. The set of minterms associated by step 2 is given $\{x_1x_2x_4x_5\}$. The PBF of LUM smoother is given by following summations of minterms obtained by steps 1 and 2:

$$f_B(x_1, x_2, x_3, x_4, x_5) = x_1x_3 + x_2x_3 + x_3x_4 + x_3x_5 + x_1x_2x_4x_5. \quad (10)$$

For algorithm mentioned above, a number of minterms done by single algorithm steps can be expressed. The derivation is based on colored input set. The presence of central sample $x^*=x_{(N+1)/2}$ in minterms can be indicated by temporal coloring [2,10] of two colors, where x^* is marked by different color from neighbourhoods. On the other hand, spatial colorings of two colors can represent all created minterms of k or N-k+1 elements. Finally, the resultant minterms are obtained by the combination of colored vectors given by booth spatial and temporal colorings, thus, creating the spatiotemporal colored vectors.

4.1 Step 1

Minterms of k elements with the presence of central sample $x_{(N+1)/2}$ can be expressed by group of permutations, that is reduced by coloring methods [1,2,10].

Let input set of N elements $\Omega = \{1,2,...,N\}$. Thus, a group of permutations is a mapping $\Omega \rightarrow \Omega$. A number of all possible permutations is given by N! The minterms of k elements can be expressed as a group of spatial two-colored permutations, where $1 \le k \le N$, the first color indices k elements, by the second color N-k elements are marked.

Thus, a number of spatial colored (SC) permutations [1,2,10] is equivalent to number of minterms of k elements that is given by

$$\left|\Omega_{S1}\right| = \frac{N!}{k!(N-k)!} \quad (11)$$

where Ω_{S1} is the group of spatial colored permutations, N is a window size and k is tuning parameter of LUM smoothers. However, according to step 1, the minterms must contain the central sample $x_{(N+1)/2}$. From above, the presence of $x_{(N+1)/2}$ is characterised by temporal coloring (TC). Thus, minterms of k elements with the presence of central sample $x_{(N+1)/2}$ must correspond to the expression spatiotemporal colored (STC) permutations (Fig. 2). Number of minterms according to step 1 is the same to the number of STC vectors given by

$$\left|\Omega_{ST1}\right| = \left|\Omega_{SP1}\right|\frac{k}{N} = \frac{N!}{k!(N-k)!}\frac{k}{N} = \frac{(N-1)!}{(k-1)!(N-k)!} \quad (12)$$

Proof: Since, each element of input set W is frequently represented just others, the number of all elements from minterms obtained by SC (11) is simply $k|\Omega_{S1}|$. Then occurrence of arbitrary element $x_1, x_2, ..., x_N$ is identical and given by $k|\Omega_{S1}|/N$. It is equal to occurrence of $x_{(N+1)/2}$ that is a number of k-elements minterms with $x_{(N+1)/2}$. □

Example 2: Consider a identical conditions to Example 1. The number of minterms of 2 elements that contain central sample x_3 is given by (5-1)!/((2-1)!(5-2)!)=4. This result is corresponding to (10).

4.2 Step 2

Now, create the minterms of N-k+1 elements without central sample $x_{(N+1)/2}$ (Fig. 2). All minterms of N-k+1 elements are expressed as the set of SC vectors. Thus, the number of these SC permutations is given by

$$\left|\Omega_{S2}\right| = \frac{N!}{(N-k+1)!(N-(N-k+1))!} = \frac{N!}{(N-k+1)!(k-1)!} \quad (13)$$

where variables are identical to (11). Absence of central sample is presented by TC of two colors, however, with opposite colors to TC in step 1. Since, a number of minterms corresponding to step 2 is expressed by (13) multiplication of $|\Omega_{S2}|$ and factor $(k-1)/N$.

Proof: Number of SC vectors is $|\Omega_{S2}|$. Colored vectors include N elements, then $|\Omega_{S2}|/N$ colored vectors correspond to each element. Since, $x_{(N+1)/2}$ is not used, a number of useful parameters is k-1. Then, a number of STC vectors given by step 2 is $(k-1)|\Omega_{S2}|/N$. □

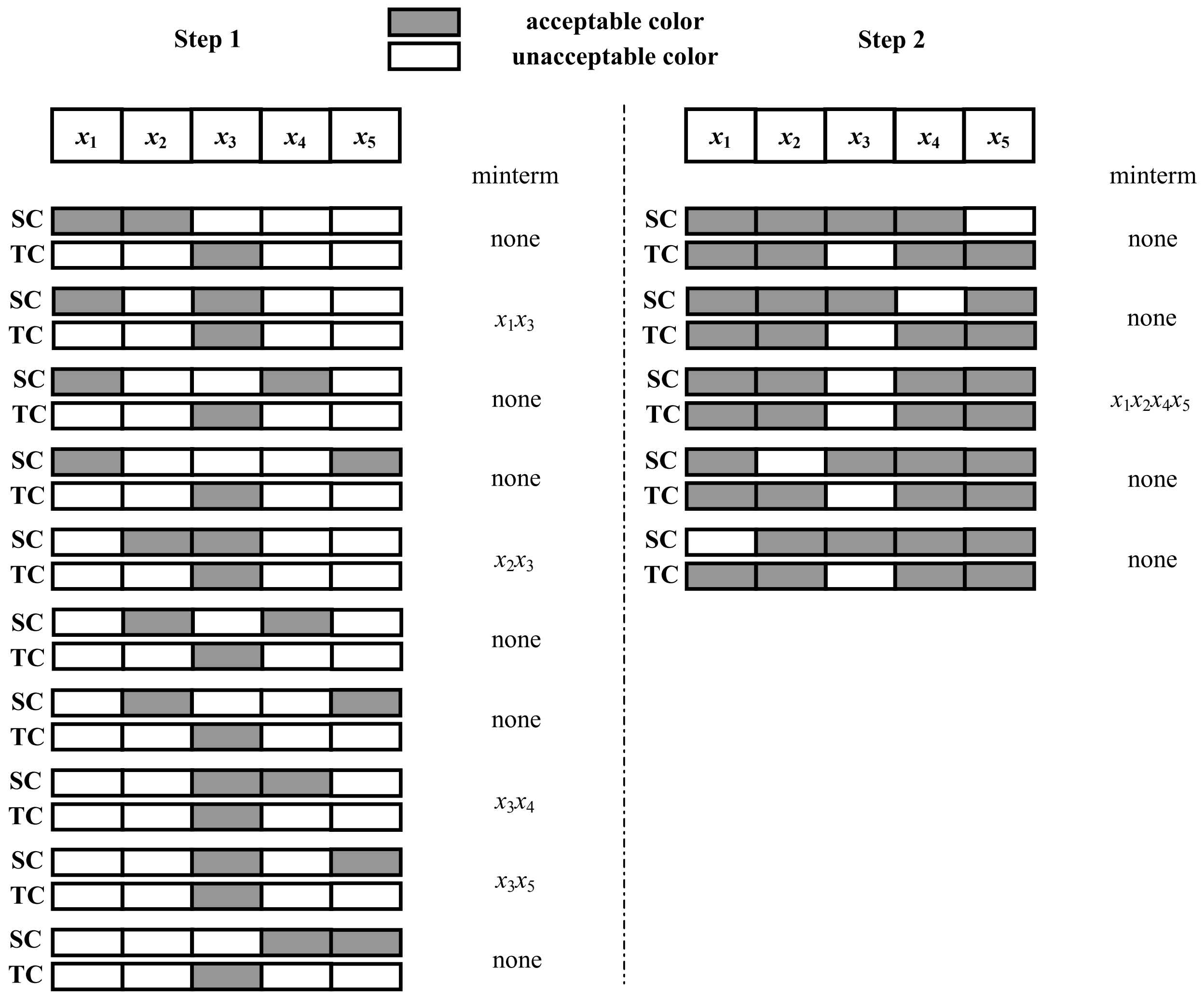

Fig. 2 Spatiotemporal coloring

N	5		9		11		13		25		27	
k	*ST1*	*ST2*	*ST1*	*ST2*	*ST1*	*ST2*	*ST1*	*ST2*	*ST1*	*ST2*	*ST1*	*ST2*
1	1	0	1	0	1	0	1	0	1	0	1	0
2	4	1	8	1	10	1	12	1	24	1	26	1
3	6	4	28	8	45	10	66	12	276	24	325	26
4			56	28	120	45	220	66	2024	276	2600	325
5			70	56	210	120	495	220	10626	2024	14950	2600
6					252	210	792	495	42504	10626	65780	14950
7							924	792	134596	42504	230230	65780
8									346104	134596	657800	230230
9									735471	346104	1562275	657800
10									1307504	735471	3124550	1562275
11									1961256	1307504	5311735	3124550
12									2496144	1961256	7726160	5311735
13									2704156	2496144	9657700	7726160
14											10400600	9657700

Tab. 1 LUM smoother – a number of minterms obtained by step 1 (ST1) and step 2 (ST2) for various window size N.

Thus, a number of minterms according to step 2 is given by

$$\begin{aligned}\left|\Omega_{ST2}\right| &= \left|\Omega_{RP2}\right|\frac{k-1}{N}\\ &= \frac{N!}{(N-k+1)!(k-1)!}\frac{k-1}{N}\end{aligned} \quad (14)$$

After the reduction, (14) can be expressed as

$$\left|\Omega_{ST2}\right| = \begin{cases} 0 & \text{for } k=1 \\ \dfrac{(N-1)!}{(N-k+1)!(k-2)!} & \text{otherwise} \end{cases} \quad (15)$$

Equation (15) include necessary and sufficient condition for identity filter, i.e. k=1 and the filter output is identical to central sample $x_{(N+1)/2}$, that is satisfied by step 1. Thus, by the step 2 cannot be obtained next minterm.

Proof: According to the condition of step 2, all minterms are created by N-k+1 elements without $x_{(N+1)/2}$. Clearly, for k=1, minterms satisfied step 2 must contain N-1+1=N elements $x_1x_2...x_N$, that is possible with the presence of $x_{(N+1)/2}$, only. Thus, from the N-inputs set is not possible to create minterm of N elements without $x_{(N+1)/2}$. For k=1, the number of minterms without central sample is equal to 0. This results is derived from (14), where the limited factor k-1 is considered. □

Example 3: Consider identical conditions to Example 1. The number of minterms of 5-2+1=4 elements without presence of central sample x_3 is given by (5-1)!/((5-2+1)!(2-2)!)=1. This result is corresponding to (10).

5. Conclusion

In this paper, the coloring concept of Boolean LUM smoothers was presented. Thus, the minimal PBF for each tuning level done by LUM smoothers was obtained directly. In addition, through permutation theory, temporal and spatial colorings a number of minterms was derived and proved (Table 1). Now, the design of Boolean LUM smoother is more simplified and faster.

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