IDENTIFYING THE DETERMINISTIC CHAOS BY USING THE LYAPUNOV EXPONENTS

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Abstract

This paper presents an investigation of deterministic chaos. The modified Colpitts oscillator is used as an example of deterministic chaos in electronic circuits. Most known quality of deterministic chaotic systems the positive Lyapunov exponent is used for investigation the chaotic Colpitts oscillator.

Keywords

Deterministic chaos, Lyapunov exponents, Colpitts oscillator

1. Introduction

In [2] the recurrent quality of deterministic chaotic circuits were described by using Lorenz maps. This recurrent quality also causes chaotic looks and the very important property of sensitivity with respect to the initial conditions also known as positive Lyapunov exponent. This attribute was at first mentioned by meteorologist Lorenz. He compared the high sensitivity of deterministic chaotic systems by the unpredictablility of the weather forecast in a long time limit. In other words any little change in initial conditions causes fast divergence in two same trajectories in phase space, which can be measured by assuming the Lyapunov exponents of investigated system.

2. The Largest Lyapunov Exponent

Consider a dynamical system defined by a set of ordinary differential equations

$$\dot{x} = f(x) \quad , \quad x(0) = x_{(0)} . \tag{1}$$

Let us take 2 neighbouring points $x_1(0)$ and $x_2(0)$ in phase space and observe the behaviour of trajectories originating at $t = 0$ from these two points. In particular, we can measure the separation $\delta(t) = | x_2(t) - x_1(t) |$ between these trajectories at any time moment t. Then, if the dynamical system (1) is deterministic chaotic the $\delta(t)$ increases exponentially with time, so that in a long time limit we get

$$\delta(t) \approx \delta(0).e^{\lambda t} . \tag{2}$$

Using (2) we can estimate the approximate value of largest Lyapunov exponent

$$\lambda \approx \frac{1}{t}\ln\left|\frac{\delta(t)}{\delta(0)}\right| . \tag{3}$$

This estimate is not, however acceptable, because the distance $\delta(t)$ cannot grow indefinitely. Therefore, at large times λ approach zero for all system's family. However, the smaller is the initial separation $\delta(0)$ the longer we can observe the growth of $\delta(t)$ before it reaches its maximal value. So we can then rewrite (3) in more acceptable form

$$\lambda = \lim_{\substack{d(0)\to 0 \\ t\to\infty}} \frac{1}{t}\ln\left|\frac{\delta(t)}{\delta(0)}\right| . \tag{4}$$

Using (4) we can assume the largest Lyapunov exponent. For three-dimensional system, the largest Lyapunov exponent is the positive Lyapunov exponent and in this case λ is also equal to the Kolmogorov-Sinai (KS) entropy, which is defined such that

$$E_{KS} \le \sum_i \lambda_i^+ , \tag{5}$$

where the $\{\lambda_i^+\}$ are positive Lyapunov exponents. The largest Lyapunov exponent λ is important property of dynamical systems, and its positive value is the sufficient condition for deterministic chaotic systems.

The analytical calculation of Lyapunov exponent is usually not possible. Nevertheless, there are many numerical algorithms, which allow numerical computation of the largest Lyapunov exponent. Equation (4) in discrete form can be also readily used for computation of λ.

3. Global Lyapunov Exponents

However, the positive (largest) Lyapunov exponent is the most important for deterministic chaotic systems, there are also other exponents that characterize the dynamical systems. The total number of exponents is equal to the dimensionality n of the phase space of a particular dynamical system. All deterministic chaotic systems have at least one positive Lyapunov exponent. If a trajectory does not end in a fixed point, one of the Lyapunov exponents is always equal to zero. Finally, if the system's parameters do not exceed some concrete limits (the system is dissipative), the sum of the Lyapunov exponents must be less than zero, so there must be for deterministic chaotic system also at least one Lyapunov exponent smaller then zero. In general we can summarize:

$\lambda_1 \geq \ldots \geq \lambda_n$

$\lambda_i < 0, i = 1,\ldots,n \quad \Rightarrow$ stable equilibrium

$\lambda_1 = 0, \lambda_i < 0, i = 2,\ldots,n \quad \Rightarrow$ stable limit cycle

$\lambda_1 = \lambda_2 = 0, \lambda_i < 0, i = 3,\ldots,n \quad \Rightarrow$ stable two-torus

$\lambda_1 = \ldots = \lambda_m = 0, \lambda_1 < 0, i = m+1,\ldots,n \quad \Rightarrow$ stable m-torus

For three-dimensional time-continuous autonomous system, the only possibility for chaos to exist is that the three Lyapunov exponents are

$\lambda_1 > 0, \lambda_2 = 0, \lambda_2 < 0$ and $\lambda_3 < -\lambda_1$

For four-dimensional time-continuous systems, there are only three possibilities for chaos to emerge:

$\lambda_1 > 0, \lambda_2 = 0, \lambda_4 \leq \lambda_3 < 0$; leading to chaos

$\lambda_1 \geq \lambda_2 > 0, \lambda_3 = 0, \lambda_4 < 0$; leading to 'hyperchaos'

$\lambda_1 > 0, \lambda_2 = \lambda_3 = 0, \lambda_4 < 0$; leading to 'chaotic two-torus'

(this special orbit has not been experimentally observed).

By Loskutov and Mikhailov [3] we can define the global Lyapunov exponents as follows:

Let us take a trajectory $x(t)$ of a dynamical system (1), originating from the point $x(0)$, and another neighbouring trajectory $x_1(t) = x(t) + \xi(t)$. Consider the function

$$\Lambda(\xi(0)) = \lim \frac{1}{t} \ln\left[\frac{|\xi(t)|}{|\xi(0)|}\right] \tag{6}$$

defined for the vectors $\xi(0)$ of the initial shift, such that $|\xi(0)| = \varepsilon$ and $\varepsilon \to 0$. We claim that, under all possible rotations of the vector $\xi(0)$, the function Λ will change by jumps and will acquire a finite set of values $\{\lambda_j\}, j = 1, 2, \ldots, n$. These values are called the (global) Lyapunov exponents. Most precisely is the determining of Ljapunov exponents from a time series described in [6,7].

4. The Practical Utilization of Lyapunov exponents

As we have seen the three-dimensional systems such as chaotic Colpitts oscillator have three Lyapunov exponents, where the first is positive the second is equal to zero and third, last is negative. The most important Lyapunov exponent is the largest (positive) because in dissipative systems only for deterministic chaotic one it exists. Now, it is obvious that the presence of positive Lyapunov exponent in dissipative system prove the deterministic chaotic behaviour. And also hold that the largest Lyapunov exponent as well as the positive Lyapunov one and as well as the KS entropy are equal, for all 3D chaotic systems.

The chaotic Colpitts oscillator [1] in Fig. 1 for presence of positive Lyapunov exponent, using simulation program P-SPICE, has been investigated. This oscillator has chaotic character for certain circuit's parameters (Fig. 1). In this type of oscillators, to get chaotic behaviour must be the resonant part of circuit strongly attenuated. In Kennedy's modification of Colpitts oscillator is this condition satisfied by resistor R_L in series with inductor L_1.

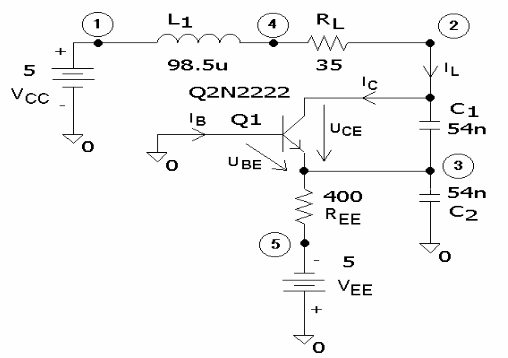

Fig. 1 Chaotic Colpitts oscillator by [1].

5. Simulations in P-SPICE

The transient simulation of Colpitts oscillator was used. The three main independent variables U_{CE}, U_{BE} and I_L were investigated. In P-SPICE simulation two identical Colpitts oscillator were used with little change only in initial condition of capacitor C_1. In Fig. 2 is investigated the variable U_{BE}. At the top is the difference of two continuances from the bottom. The distinction in initial condition (1mV) takes value comparable with the amplitude of continuance approximate at time 130 μs. This quality is also called as high sensitivity with respect to initial condition (IC) what also designate the presence of positive Lyapunov exponent.

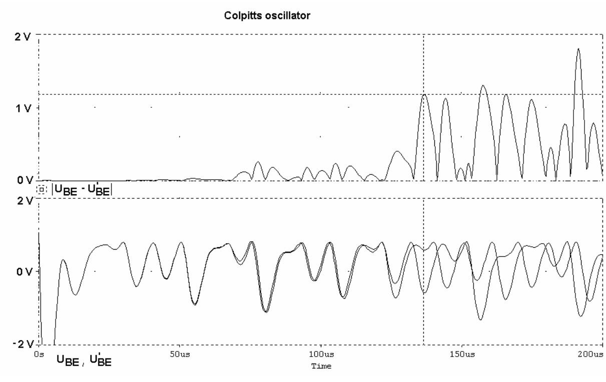

Fig. 2 Test for presence of positive Lyapunov exponent for U_{BE}. The difference in IC is 1mV. The approximate Lyapunov exponent is 40000.

The approximate value of largest Lyapunov exponent can be also computed using eq. (3). The t is equal to the time, were the difference takes values comparable with the amplitude of the continuance. The $\delta(t)$ is equal to this amplitude and the $\delta(0)$ is equal to the little change in IC. The value of largest Lyapunov exponent get using this method is only approximate.

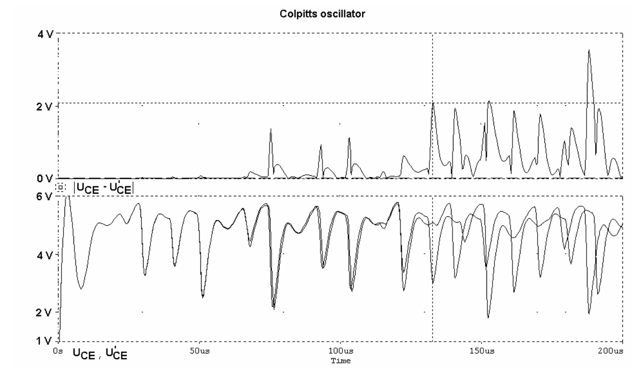

Fig. 3 Test for presence of positive Lyapunov exponent for U_{CE}. The difference in IC is 1mV. The approximate Lyapunov exponent is 60000.

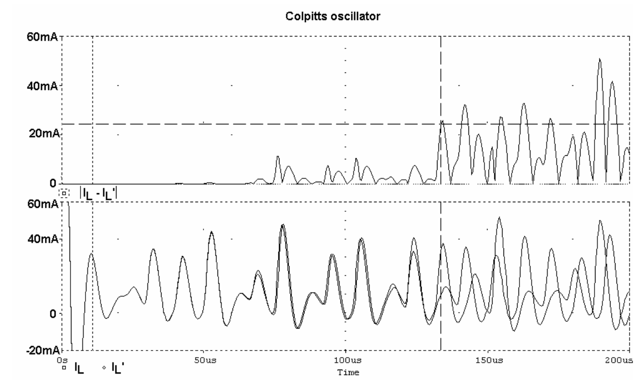

Fig. 4 Test for presence of positive Lyapunov exponent for I_L. The difference in IC is 30 μA. The approximate Lyapunov exponent is 50000.

In Fig. 3 and Fig. 4 are same tests as in Fig. 2 but for variables U_{CE} and I_L For variable I_L we estimated the IC from time continuance at the time near to zero.

This method we approved on Lorenz system [4] were we assumed also the largest Lyapunov exponent. The difference was less then ten percent from the value declared in [3].

6. Conclusion

This paper briefly discusses the notion of Lyapunov exponents in deterministic chaotic systems, especially in three-dimensional systems. We have reached the conclusion that the sufficient condition for deterministic chaotic behaviour in the three-dimensional dynamical systems is the positive largest Lyapunov exponent. We also showed how can be verified this condition using simulation program and consequently assumed the approximate value of the largest Lyapunov exponent.

For chaotic Colpitts oscillator [1] we also assumed the largest Lyapunov exponent with average value around 50000. This value in compare with other known systems (Lorenz system [4,6], $\lambda = 1.37$) seems to be large because the high frequency of Colpitts oscillator in compare with other one and therefore also with much smaller time t in eqn. (3).

In general we can summarize that, the positive Lyapunov exponent in any dissipative systems suggest the presence of deterministic chaotic behaviour. In the case if the system is three or four-dimensional, it means that the system is deterministic chaotic.

References

[1] KENNEDY, P., Chaos in the Colpitts Oscillator. IEEE Trans. Circuits Syst., vol. 41, no. 11, p. 771-774, 1994

[2] KVARDA, P., Identifying the Deterministic Chaos by Using the Lorenz Maps. Radioengineering, vol. 9, no. 4, p. 32-33, 2000

[3] MIKHAILOV, A. S., LOSKUTOV A. Y., Foundation of Synergetics II: Complex Patterns. Berlin: Springer, 1991

[4] LORENZ, E. N., J. Atmos. Sci., vol. 20, p. 130, 1963

[5] WEBSTER, J., Wiley Encyclopedia of Electrical and Electronics Engineering Online, John Wiley & Sons, (http://www.interscience.wiley.com). 1999

[6] SHIMADA I., NAGASHIMA T., A Numerical Approach to Ergodic Problem of Dissipative Dymamical Systems. Progress of Theoretical Physics, vol. 61, no. 6, June 1979

[7] WOLF A., SWIFT B., Determining Lyapunov Exponents from a Time Series. Physica, vol. 16D, p. 285-317, 1985

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