SYNTHESIS OF OPTIMIZED PIECEWISE-LINEAR SYSTEMS USING SIMILARITY TRANSFORMATION
PART II: SECOND-ORDER SYSTEMS

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Abstract

State models of dynamical systems can be used as prototypes in practical realization of electronic chaotic oscillators. Experimental verification shows namely eigenvalue sensitivities of these prototypes are very important for such a purpose. In the paper the optimization design procedure for the second-order linear and piecewise-linear (PWL) autonomous dynamical systems is suggested. This gives the possibility to obtain minimum eigenvalue sensitivities with respect to the change of the individual state model parameters.

Keywords

Dynamical systems, piecewise-linear systems, sensitivity, optimization

1. Introduction

Starting from the current simple models topologically conjugate to Class *C* [1] - [4], the resultant optimized models can be synthesized using the general topological conjugacy conditions [5]. The original and the resultant systems are qualitatively equivalent and can be described by the general state matrix form as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\, h\,(\mathbf{w}^{\mathrm{T}}\mathbf{x}) \tag{1a}$$

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{b}}\, h\,(\tilde{\mathbf{w}}^{\mathrm{T}}\tilde{\mathbf{x}}) \; , \tag{1b}$$

where elementary PWL feedback functions

$$h\,(\mathbf{w}^{\mathrm{T}}\mathbf{x}) = \frac{1}{2}\left(\left|\mathbf{w}^{\mathrm{T}}\mathbf{x}+1\right| - \left|\mathbf{w}^{\mathrm{T}}\mathbf{x}-1\right|\right) \tag{2a}$$

$$h\,(\tilde{\mathbf{w}}^{\mathrm{T}}\tilde{\mathbf{x}}) = \frac{1}{2}\left(\left|\tilde{\mathbf{w}}^{\mathrm{T}}\tilde{\mathbf{x}}+1\right| - \left|\tilde{\mathbf{w}}^{\mathrm{T}}\tilde{\mathbf{x}}-1\right|\right) \tag{2b}$$

determine the regions D_0 and D_{+1} (D_{-1}), respectively [5]. The dynamical behavior of both these systems is determined by two characteristic polynomials associated to the individual regions, i.e.

$$D_0\text{:}\quad P(s) = (s-\mu_1)(s-\mu_2)\;\ldots\;(s-\mu_n) = \det(s\mathbf{1}-\mathbf{A}_0) = \\ = s^3 - p_1 s^{n-1} + p_2 s^{n-2} - \ldots + (-1)^{n+1} p_{n-1} s + (-1)^n p_n \tag{3}$$

$$D_{\pm 1}\text{:}\quad Q(s) = (s-\nu_1)(s-\nu_2)\;\ldots\;(s-\nu_n) = \det(s\mathbf{1}-\mathbf{A}) = \\ = s^3 - q_1 s^{n-1} + q_2 s^{n-2} - \ldots + (-1)^{n+1} q_{n-1} s + (-1)^n q_n \tag{4}$$

where **1** is the unity matrix. Its roots represent the eigenvalues of the state matrices $\mathbf{A}_0$, $\mathbf{A}$, the coefficients are called the equivalent eigenvalue parameters [4], [5].

Mutual relation between these two systems can be expressed by the general linear transformation [5]

$$\tilde{\mathbf{x}} = \mathbf{T}\,\mathbf{x}, \quad \tilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \quad \tilde{\mathbf{b}} = \mathbf{T}\mathbf{b} \tag{5a,b,c}$$

where the transformation matrix **T** is to be optimized from the viewpoint of the minimum sum of relative eigenvalue sensitivity squares with respect to the change of the individual state matrix parameters.

2. Second-Order Systems

In case of the second-order systems the optimization condition is obtained in the general symbolical form [6]

$$(a_{11}-a_{22})(t_{11}\,t_{22}+t_{21}\,t_{12}) = 2(t_{11}\,t_{21}\,a_{12} - t_{12}\,t_{22}\,a_{21}) \tag{6}$$

where a_{ij} and k_{kl} are the individual parameters of the state matrix **A** and the transformation matrix **T**, respectively. Special form of this condition and the related optimized resultant state matrices $\mathbf{A}^{\sim}$ and vectors $\mathbf{b}^{\sim}$ have been derived for several types of initial models, namely for the elementary Jordan form (in case of the real eigenvalues), decomposed complex form [2] (in case of the complex conjugate eigenvalues), and also both canonical forms [1]. In all these models derived the minimum sum of relative eigenvalue sensitivity squares is obtained as the optimum case. As the PWL approach is used all these systems can be considered linear in each of the regions. Then the eigenvalues designated generally λ_i (i = 1, 2) represent μ_i or ν_i in the region D_0 or D_{+1} (D_{-1}), respectively.

2.1 Real eigenvalues

Starting from the elementary Jordan form of the initial state matrix

$$\mathbf{A}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \tag{7}$$

the optimization condition (6) can be simplified as

$$t_{11}\,t_{22}+t_{21}\,t_{12}=0 \tag{8}$$

and then the resultant state matrix of the optimized model is

$$\widetilde{\mathbf{A}}=\frac{1}{2}\begin{bmatrix}(\lambda_1+\lambda_2) & (\lambda_1-\lambda_2)K\\ (\lambda_1-\lambda_2)K^{-1} & (\lambda_1+\lambda_2)\end{bmatrix} \tag{9}$$

where

$$K=\frac{t_{12}}{t_{22}}=-\frac{t_{11}}{t_{21}} \tag{10}$$

The optimum values of the sensitivity measure are

$$\sum_{i,j}S_r^2(\lambda_1,\widetilde{a}_{ij})=\frac{1}{4}\left[1+\left(\frac{\lambda_2}{\lambda_1}\right)^2\right] \tag{11a}$$

$$\sum_{i,j}S_r^2(\lambda_2,\widetilde{a}_{ij})=\frac{1}{4}\left[1+\left(\frac{\lambda_1}{\lambda_2}\right)^2\right] \tag{11b}$$

and evidently do not depend on the coefficient K. It can be used for the complete state model design having the optimum form in all regions of PWL function.

2.2 Complex conjugate eigenvalues

In this case $\lambda_{1,2}=\lambda'\pm j\lambda''$, the initial state matrix is considered in the decomposed complex form [2]

$$\mathbf{A}=\begin{bmatrix}\lambda' & -\lambda''\\ \lambda'' & \lambda'\end{bmatrix} \tag{12}$$

and the optimization condition (7) is modified as

$$t_{11}\,t_{21}+t_{22}\,t_{12}=0\ . \tag{13}$$

Then the resultant state matrix of the optimized model is very similar, i.e.

$$\mathbf{A}=\begin{bmatrix}\lambda' & -\lambda''K\\ \lambda''K^{-1} & \lambda'\end{bmatrix} \tag{14}$$

where

$$K=\frac{t_{12}}{t_{21}}=-\frac{t_{11}}{t_{22}} \tag{15}$$

so that the initial state matrix (12) represents the special case of the optimum form (15) for $K=1$, with the same optimum sensitivities

$$\sum_{i,j}S_r^2(\lambda',\widetilde{a}_{ij})=\frac{1}{2} \tag{16a}$$

$$\sum_{i,j}S_r^2(\lambda'',\widetilde{a}_{ij})=\frac{1}{2} \tag{16b}$$

2.3 Application in PWL systems

Consider the most often case in oscillators, i.e. the complex conjugate eigenvalues in the outer regions D_{+1}, D_{-1} ($\nu_{1,2}=\nu'\pm j\nu''$), as in the inner region D_0 ($\mu_{1,2}=\mu'\pm j\mu''$). The optimized state matrix corresponding to the outer regions can be chosen in the simplified decomposed complex form (14) - $K=1$, i.e.

$$\widetilde{\mathbf{A}}=\begin{bmatrix}\nu' & -\nu''\\ \nu'' & \nu'\end{bmatrix} \tag{17}$$

while the state matrix for the inner region is considered in the general form

$$\widetilde{\mathbf{A}}_0=\begin{bmatrix}\mu' & -\mu''K\\ \mu''K^{-1} & \mu'\end{bmatrix} \tag{18}$$

These state matrices are mutually related [4] as follows

$$\widetilde{\mathbf{A}}_0=\widetilde{\mathbf{A}}+\widetilde{\mathbf{b}}\,\mathbf{w}^{\mathrm{T}} \tag{19}$$

where

$$\widetilde{\mathbf{b}}=[\widetilde{b}_1\quad \widetilde{b}_2]^{\mathrm{T}}\quad\text{and}\quad \widetilde{\mathbf{w}}=[\widetilde{w}_1\quad \widetilde{w}_2]^{\mathrm{T}}\ ,$$

so that in detail

$$\widetilde{\mathbf{A}}_0=\begin{bmatrix}\nu'+\widetilde{b}_1\widetilde{w}_1 & -\nu''+\widetilde{b}_1\widetilde{w}_2\\ \nu''+\widetilde{b}_2\widetilde{w}_1 & \nu'+\widetilde{b}_2\widetilde{w}_2\end{bmatrix} \tag{20}$$

Comparing (18), (20) following formulas can be obtained

$$\widetilde{b}_1\widetilde{w}_1=\widetilde{b}_2\widetilde{w}_2=(\mu'-\nu') \tag{21a,b}$$

$$\widetilde{b}_1\widetilde{w}_2=(\nu''-\mu''K) \tag{21c}$$

$$\widetilde{b}_2\widetilde{w}_1=(\mu''K^{-1}-\nu'') \tag{21d}$$

and then rewritten into the form

$$\frac{\widetilde{b}_1}{\widetilde{b}_2}=\frac{\widetilde{w}_2}{\widetilde{w}_1}=\frac{\nu''-\mu''K}{\mu'-\nu'}=\frac{(\mu'-\nu')K}{\mu''-\nu''K} \tag{22}$$

The optimizing coefficient K can finally be expressed as the real root of the quadratic equation

$$K^2-2K(M+1)+1=0 \tag{23a}$$

$$K_{1,2}=1+M\pm\sqrt{M(M+2)} \tag{23b}$$

where the auxiliary parameter M is given in the form

$$M=\frac{(\mu'-\nu')^2+(\mu''-\nu'')^2}{2\mu''\nu''}>0,\quad (\mu'',\nu''\neq 0) \tag{23c}$$

Choosing $w_1^{\sim}=1$, the other parameters are obtained as

$$\widetilde{b}_1=\mu'-\nu',\quad \widetilde{b}_2=\frac{(\mu'-\nu')^2}{\nu''-\mu''K},\quad \widetilde{w}_2=\frac{\nu''-\mu''K}{\mu'-\nu'} \tag{24}$$

Denoting $\mathbf{x}^{\sim}=[x_1^{\sim}\ x_2^{\sim}]^{\mathrm{T}}$, the complete state equations of the optimized second-order PWL autonomous system can be written in the form

$$\dot{\tilde{x}}_1 = \nu'\left[\tilde{x}_1 - h\,(\tilde{x}_1 + \tilde{w}_2\,\tilde{x}_2)\right] - \nu''\,\tilde{x}_2 + \mu'\,h\,(\tilde{x}_1 + \tilde{w}_2\,\tilde{x}_2) \quad (25)$$

$$\dot{\tilde{x}}_2 = \nu''\,\tilde{x}_1 + \nu'\,\tilde{x}_2 + \tilde{b}_2\,h\,(\tilde{x}_1 + \tilde{w}_2\,\tilde{x}_2) \quad (26)$$

where the parameters $b_2^{\sim}$, $w_2^{\sim}$ are given by (24). The corresponding integrator-based circuit block diagram, suitable as the prototype for the practical realization, is shown in Fig. 1 (without "wave" designation).

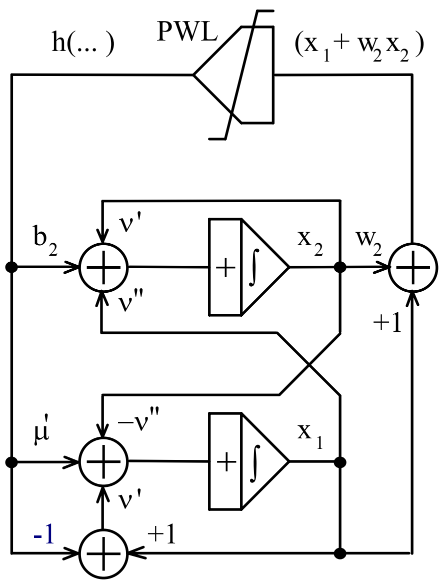

Fig. 1 Integrator-based circuit structure of the 2nd-order state model with minimized sensitivities.

3. Conclusion

General optimization condition for the second-order PWL dynamical systems is introduced. It enables us to design their state models with low eigenvalue sensitivities and then applied them also for synthesis of the optimized higher-order systems [7]. The results achieved are important for the practical realization of chaotic dynamical systems in the form of simple electronic circuits having separately adjustable parameters. It has been also proved by first laboratory experiments.

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References

[1] HANUS, S. Realization of Third-Order Chaotic Systems Using Their Elementary Canonical State Models. In Proc. of the conference Rádioelektronika'97. Bratislava, 1997, p. 44-45.

[2] POSPÍŠIL, J., BRZOBOHATÝ, J. Elementary Canonical State Models of Chua's Circuit Family. IEEE Trans. Circ. Syst.-I: Fundamentals. 1996, vol. 43, no. 8, p. 702-705.

[3] POSPÍŠIL, J., BRZOBOHATÝ, J., KOLKA, Z., HORSKÁ, J. Decomposed Canonical State Models of the Third-Order Piecewise-Linear Dynamical Systems. In Proc. ECCTD'99, Stresa, 1999, p. 181-184.

[4] POSPÍŠIL, J., BRZOBOHATÝ, J., KOLKA, Z., HORSKÁ, J. Simplest ODE Equivalents of Chua's Equations. International Journal of Bifurcation and Chaos. 2000, vol. 10, no. 1, p. 1-23.

[5] CHUA, L.O., WU, C.W. On Linear Topological Conjugacy of Lure's Systems. IEEE Trans. CAS, 1996, vol. 43, p. 158-161.

[6] KOLKA, Z. Using Similarity Transformation for Nonlinear System Synthesis. In: Proc. Radioelektronika '2001. Brno, 2001, p. 5-7.

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