# Improvements of Analog Neural Networks Based on Kalman Filter

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## Abstract

*In the paper, original improvements of recurrent analog neural networks, which are based on Kalman filter, are presented. These improvements eliminate some disadvantages of the classical Kalman neural network and enable a real time processing of quickly changing signals, which appear in adaptive antennas and similar applications. This goal is reached using such circuit elements, which increase the convergence rate of the network and decrease the dependence of convergence rate on the ratio of eigenvalues of the correlation matrix of input signals.*

## Keywords

Kalman filter, analog recurrent neural networks, convergence rate, stability

## 1. Introduction

Artificial neural networks can be described as parallel non-linear electronic systems, which exhibit the learning ability. Since parallel multi-processor systems are rather expensive, analog versions of neural networks can be very advantageous in some kinds of applications (adaptive antennas, real-time optimization, etc.).

The learning ability of neural networks is realized using various adaptive algorithms (Least Mean Squares [6], e.g.). In our development, the Kalman filter is used for training due to its higher convergence rate. On the other hand, Kalman neural nets are rather complicated, which increases complexity of the analog circuitry. Nevertheless, the circuitry can be simplified without any significant degradation of convergence properties [1].

In our paper, we concentrate on various improvements of the simplified Kalman neural network (SKN), which increase the convergence rate and decrease dependency of the convergence properties to the statistical parameters of input signals.

## 2. Kalman neural network

Kalman neural networks remove negative features of LMS-trained neural nets (especially, slow convergence rate is increased) [1]. The Kalman neural network for solving of matrix equation $\mathbf{A}\ \mathbf{v}_S = \mathbf{b}$ can be described by the following set of equations [1]:

$$d\mathbf{v}(t)/dt = \mathbf{K}(t)[\mathbf{b} - \mathbf{A}.\mathbf{v}(t)], \tag{1}$$

$$\mathbf{K}(t) = \mathbf{P}(t)\,\mathbf{A}^{\mathrm{T}}\,\mathbf{R}^{-1}, \tag{2}$$

$$d\mathbf{P}(t)/dt = -\mathbf{K}(t)\,\mathbf{A}\,\mathbf{P}(t), \tag{3}$$

where $\mathbf{P}(t)$ is a predicted state-error, $\mathbf{R}^{-1}$ is inverted correlation matrix of a residual error, $\mathbf{K}$ denotes a matrix of a Kalman gain and $\mathbf{v}$ is a vector of solutions, which for time approaching infinity converges to the vector $\mathbf{v}_S$. Since pure Kalman net is rather complicated, a simplified version was developed [1]. The simplification is based on the exploitation of diagonal elements of the matrix $\mathbf{P}(t)$ only.

In analog implementation, simplifications lead to decreasing the number of circuit elements. The convergence properties of the simplified Kalman net (SKN) were discussed in [2] resulting to the conclusion that convergence properties of SKN are limited by the pure Kalman net on one hand and by the LMS-based net on the other hand.

Unfortunately, the mathematical description of SKN is complicated. Therefore, we use matrix equations for pure Kalman network only (which suit in 1D case for SKN too).

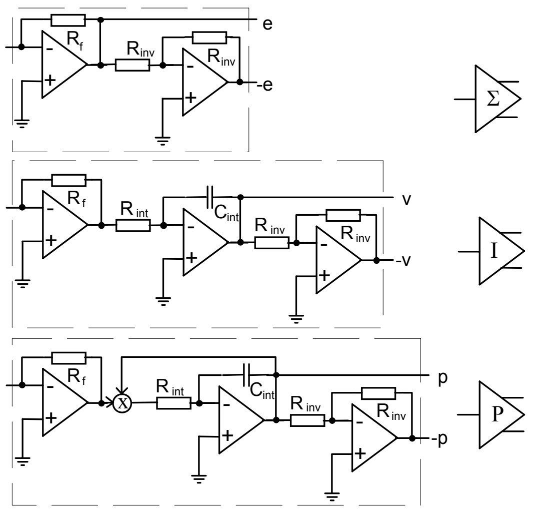


**Fig. 1** Neurons of SKN: a) summer, b) integrator, c) predictor

The SKN is depicted in Fig. 2. The network consists of three types of neurons (Fig. 1): summers, integrators and predictors (they compute diagonal elements of the predicted state-error correlation matrix **P**). In predictor, ⊗ denotes an analog multiplier. The Kalman gain is led to integrators via JFETs, which serve here as electrically controlled resistors. Bias points of JFETs are set by DC sources $\theta_m$. Elements of matrix **A** are implemented in circuitry by resistors $R_{m,n}$ and $R'_{m,n}$ [1]

$$R_{m,n} = R_f / a_{m,n}, \tag{4}$$

$$R'_{m,n} = R_f / \rho a_{m,n}, \tag{5}$$

where $R_f$ is a feedback resistor of an input opamp of respective neurons, $a_{m,n}$ is an element of the matrix **A**, and [1]

$$\rho = R^{-1} > 0 . \tag{6}$$

If $R < 0$ ($R > 0$), the respective negative (positive) neuron output is used. If $a_{m,n} < 0$, then the negative output of the $k_{m,n}$ neuron is led to the gate of the JFET and the negative output of the $e_m$ neuron is led to its drain. The inverted output ($-k_{m,n}$) is connected with the input of the predictor through the resistor $R_{m,n}$ (Fig. 2). Resistors and sources of biasing thresholds of neurons are associated by the relation

$$b_m = \theta_m R_f / R_m , \tag{7}$$

where $b$ is an element of the vector **b** and $R_m$ is the resistance between a voltage source and a summer. JFETs are serving as controlled resistors and a resistor $R_{new}$ ensures stability when the resistance of JFET is small.

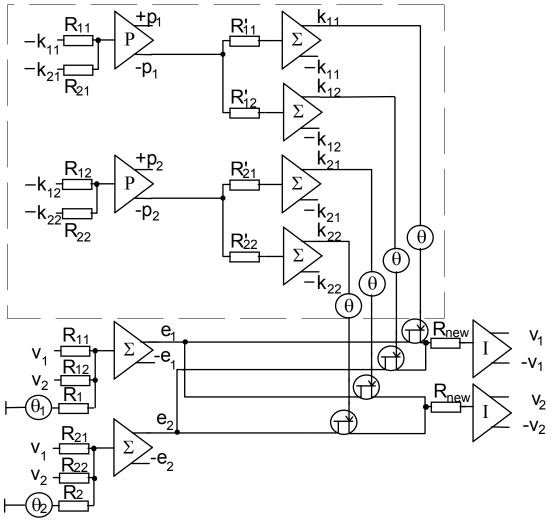

**Fig. 2** Analog realization of the SKN for solving simultaneous linear equations

# 3. Analysis of the SKN

The SKN suffers from stability problems, which are caused by the high gain of its closed loops. In 2D network, there are six closed loops (Fig. 2). The gain of each closed loop is influenced by elements of the matrix **A** (represented by $R_{m,n}$ and $R'_{m,n}$), by the Kalman gain and by the adaptive parameters of the network (time constants $R_{int}C_{int}$ in both parts of the circuit, parameter $\rho$, gains of summers and the initial voltage of $C_{int}$ in the upper part of the network). Hence, all parameters can influence stability of the system.

Performing an analysis of the network without considering real properties of circuit elements, ideal properties of the SKN are obtained. On the basis of this analysis, the influence of real properties of network components can be discussed. Therefore, the ideal 1D Kalman network is analyzed first (in this case, both SKN and in pure Kalman network are described by the same relations).

The predicted-state errors in the circuit on Fig. 2 is described by following relations

$$dP_1(t)/dt = \left(P_1^2 a_{11}^2 + P_1 P_2 a_{21}^2\right) \rho / \left(C_{int} R_{int}\right), \tag{8}$$

$$dP_2(t)/dt = \left(P_2^2 a_{22}^2 + P_1 P_2 a_{12}^2\right) \rho / C_{int} R_{int} . \tag{9}$$

The analysis of Kalman nets is quite complicated. Therefore, some simplifications are needed: the feedback resistors in summers, and time constants in predictors and integrators are considered to be identical in the whole circuitry. Next, JFETs together with $R_{new}$ and following summers in the integrator are considered being represented by the product of Kalman gain and error signal. Then, the predicted state error can be described by

$$P(t) = -\int_0^T \frac{P(t)K(t)R_f}{R_{11}C_{int}R_{int}} dt + U_0 . \tag{10}$$

Substituting Kalman gain from (2) to (10), we get

$$P(t) = -\int_0^T \frac{P^2(t)R_f^2}{R_{11}R'_{11}C_{int}R_{int}} dt + U_0 . \tag{11}$$

Differentiating of both sides of (11) yields

$$\frac{dP(t)}{dt} = -\frac{P^2(t)\, R_f^2}{R_{11}R'_{11}R_{int}C_{int}}, \tag{12}$$

which can be turned to

$$\int \frac{dP(t)}{P^2(t)} + \xi = -\int \frac{R_f^2\, dt}{R_{11}R'_{11}R_{int}C_{int}} . \tag{13}$$

Solving (13), eqn. for the predicted state error is obtained

$$P(t) = \frac{R_{11}R'_{11}C_{int}R_{int}}{R_f^2 t + \xi\, R_{int}C_{int}R_{11}R'_{11}} . \tag{14}$$

The constant $\xi$ can be evaluated from (11)

$$\xi = 1/U_0 , \tag{15}$$

and (14) becomes to

$$P(t) = \frac{U_0 R_{11}R'_{11}C_{int}R_{int}}{U_0 R_f^2 t + R_{int}C_{int}R_{11}R'_{11}} . \tag{16}$$

Then, we can express the Kalman gain as

$$K(t) = P(t)\, R_f / R'_{11} ,$$

$$K(t) = \frac{U_0 R_{11} C_{int} R_{int} R_f}{U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11}} \,. \tag{17}$$

A sub-circuit providing solution can be described by

$$e(t) = -\frac{R_f}{C_{int} R_{int}} \int_0^T \frac{e(t)\, K(t)}{R_{11}}\, dt - \frac{\theta_1 R_f}{R_1} \,. \tag{18}$$

where $e(t)$ is an error signal (output of a summer in the sub-circuit providing the solution). As stated above, the resistor $R_{new}$ (Fig. 2) is omitted here in order to simplify the analysis. Differentiating both sides of (17) and substituting to (18) yields

$$\frac{de(t)}{dt} = -\frac{e(t)\, R_f^2\, U_0}{U_0 R_f^2 t + R_{11} R'_{11} C_{int} R_{int}} \,. \tag{19}$$

This can be rewritten to

$$\frac{de(t)}{e(t)} = -\frac{R_f^2\, U_0\, dt}{U_0\, R_f^2\, t + R_{11}\, R'_{11}\, C_{int}\, R_{int}} \,. \tag{20}$$

Integrating of both sides of (20), we can obtain

$$e(t) = \frac{\exp(-\sigma)}{U_0 R_f^2 t + R_{11} R'_{11} C_{int} R_{int}} \,. \tag{21}$$

Here, $\sigma$ is a constant, which can be evaluated using (18) and (21) when turned for $t = 0$ to

$$\frac{\exp(-\sigma)}{R_{11} R'_{11} C_{int} R_{int}} = -\frac{\theta_1 R_f}{R_1} \,. \tag{22}$$

Hence

$$\sigma = \ln \frac{-R_1}{R_f R_{11} R'_{11} C_{int} R_{int} \theta_1} \,. \tag{23}$$

Then, the error signal can be expressed as

$$e(t) = \frac{-R_f R_{11} R'_{11} C_{int} R_{int} \theta_1}{R_1 (U_0 R_f^2 t + R_{11} R'_{11} C_{int} R_{int})} \,. \tag{24}$$

The solution signal is given by

$$v(t) = \int \frac{e(t)\, K(t)}{C_{int}\, R_{int}}\, dt + C \,, \tag{25}$$

where $C$ is a constant. From this, $v(t)$ can be expressed as

$$v(t) = \frac{\theta_1\, R_{11}}{R_1} \left( 1 - \frac{R_{11}\, R'_{11}\, C_{int}\, R_{int}}{U_0\, R_f^2\, t + R_{11}\, R'_{11}\, C_{int}\, R_{int}} \right) . \tag{26}$$

which can be rewritten to the matrix form:

$$\mathbf{v}(t) = \left[ \mathbf{I} - \left( \mathbf{A}\,\mathbf{A}^{\mathbf{T}} \frac{U_0\, \rho\, t}{C_{int}\, R_{int}} + \mathbf{I} \right)^{-1} \right] \mathbf{A}^{-1}\ \mathbf{b} \,. \tag{27}$$

Searching for the dependence of $\mathbf{v}(t)$ on eigenvalues of the correlation matrix of $\mathbf{A}$, the matrix transform described in [3] can be used: we define variables $\mathbf{v}$'= $\mathbf{E}^{-1}\mathbf{v}$, $\mathbf{b}$'= $\mathbf{E}^{-1}\mathbf{b}$, multiply (27) by $\mathbf{E}^{-1}$ and rewrite (27) to

$$\mathbf{v}'(t) = \left[ \mathbf{I} - \left( \Lambda \frac{U_0\, \rho\, t}{C_{int}\, R_{int}} + \mathbf{I} \right)^{-1} \right] \mathbf{A}^{-1}\ \mathbf{b}' \,, \tag{28}$$

where $\Lambda$ is a diagonal matrix of eigenvalues.

Eqn. 28 represents an ideal time course of the solution signal (i.e., time course of a network consisting of ideal circuit elements). In this ideal case, the network should converge to the solution within very short time, because there are no limits in setting of adaptation parameters.

In real-world case, convergence rate is limited by properties of real circuit elements (opamps especially). Even an unstable state of the network can appear if high gain of closed loops in the circuit is reached (this can be caused by an improper value of an adaptation parameter, e.g.). Hence, the gain of closed loops can be increased (in order to speed-up convergence) carefully to eliminate oscillations and divergence. In comparison with the Wang network [6], SKN can reach higher gains of closed loops at the beginning of convergence process because gains of closed loops in the lower sub-circuit of SKN are time-dependent (due to multiplying the signal by Kalman gain). Moreover, SKN provides more ways of influencing gains than the Wang net, which allows finer tuning of convergence properties.

# 4. SKN with additional feedback

In this section, exploitation of additional feedbacks in SKN is discussed. In the lower sub-circuit, we can create a feedback similar to the Compton's one [3] by connecting outputs of summers with their inputs, or by wiring inputs of integrators with inputs of summers (Fig. 3). Second way leads to better convergence due to involving Kalman gain in new closed loops. In this case, a basic equation is

$$v(t) = \int_0^T \left[ -\frac{v(t)}{R_{11}} + \frac{\theta_1}{R_1} - \frac{dv(t)}{dt} \frac{C_{int}\, R_{int}}{R_{11}} \right] \frac{R_f\, K(t)}{C_{int}\, R_{int}}\, dt \,. \tag{29}$$

Differentiating (29), we can obtain

$$\frac{dv(t)}{dt} \left[ 1 + \frac{K(t)\, R_f}{R_{11}} \right] = -\frac{v(t)\, R_f\, K(t)}{R_{11}\, C_{int}\, R_{int}} + \frac{\theta_1\, R_f\, K(t)}{R_1\, C_{int}\, R_{int}} \,. \tag{30}$$

Eqn. 30 can be easily solved by the variation of parameters. The homogenous equation contains only first term of the right side of (30) and can be rewritten to

$$\frac{dv(t)}{v(t)} = -\frac{R_f\, K(t)\, dt}{R_{11}\, C_{int}\, R_{int}} \left[ 1 + \frac{K(t)\, R_f}{R_{11}} \right]^{-1} . \tag{31}$$

Since Kalman gain is given by (17), the solution of (31) is

$$v(t) = C\, \frac{\left[ 1 + K(t)\, R_f / R_{11} \right]^{-1}}{R_f^2\, t\, U_0 + R_{11}\, R'_{11}\, C_{int}\, R_{int}} \,, \tag{32}$$

Here, $C$ is a constant that has to be properly set in order to obtain solution of (31). For this purpose, $C$ is considered as time-dependent and is expressed from (30) as

$$C(t) = \int \frac{\theta_1\, R_f\, K(t)}{R_{int}\, C_{int}\, R_1} \left(R_f^2\, t\, U_0 + R_{11}\, R'_{11}\, C_{int}\, R_{int}\right) dt + K \cdot$$

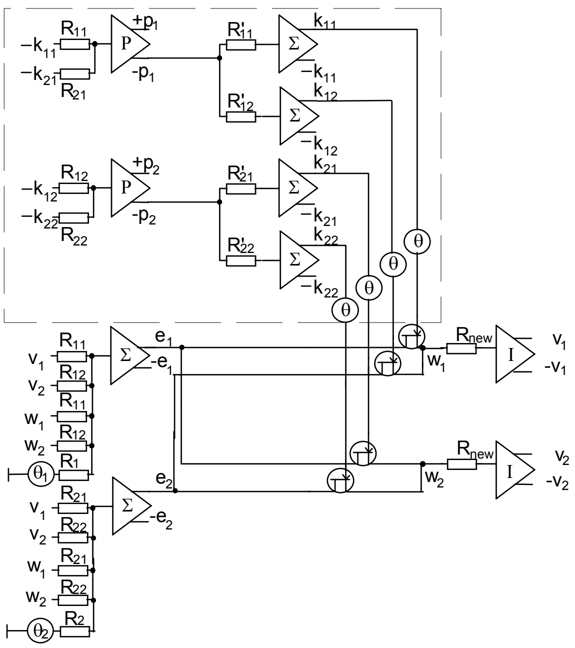


**Fig. 3** Analog realization of SKN with additional feedback

Evaluating the right side of (33), we can express $C(t)$ as

$$C(t) = \left[\theta_1\, R_f^2\, U_0\, R_{11}^2 / R_1\right] t + K\,. \tag{34}$$

Substituting (34) to (32), solution of (31) is found in form

$$v(t) = \frac{\left(\theta_1\, R_{11}\, R_f^2\, U_0\, t / R_1 + K\right)}{R_f^2\, t U_0 + C_{int}\, R_{int}\, (R_{11}\, R'_{11} + R_f^2\, U_0)}, \tag{35}$$

Here, $K$ is computed using the initial condition $v(0)=0$. That way, $K = 0$ is found and (35) can be rewritten to

$$v(t) = \frac{\theta_1\, R_{11}\, R_f^2\, U_0\, t}{R_1 \left[R_f^2\, t U_0 + C_{int}\, R_{int}\, (R_{11}\, R'_{11} + R_f^2\, U_0)\right]}, \tag{36}$$

The matrix form of (36)

$$\mathbf{v}(t) = \left[\mathbf{I} + \frac{R_{int} C_{int}}{t} \left(\mathbf{I} + \frac{1}{\rho U_0} (\mathbf{A}\mathbf{A}^T)^{-1}\right)\right]^{-1} \mathbf{A}^{-1}\, \mathbf{b}\,, \tag{37}$$

can be transformed to

$$\mathbf{v}'(t) = \left\{ \mathbf{I} - \left[\mathbf{I} + \frac{t\Lambda}{R_{int} C_{int}} \left(\Lambda + \frac{1}{\rho U_0}\right)^{-1}\right]^{-1} \right\} \mathbf{A}^{-1}\, \mathbf{b}' \tag{38}$$

using the transform [3]. Comparing the SKN with the additional feedback and the original one, a reduced influence of the eigenvalues spread to the convergence time can be observed: in (28), time is multiplied by $\Lambda\, U_0\, \rho \,/\, C_{int}\, R_{int}$, in (38), time is multiplied by $\Lambda\, \{[\Lambda + (\rho\, U_0)^{-1}]^{-1} / C_{int}\, R_{int}\}$.

# 5. SKN with modified Kalman gain

Modifying the predictor sub-circuit of SKN (the upper part of Fig. 2), the convergence rate can be increased. The modification is based on filtering Kalman gain by a low-pass filter and by using a filtered gain in the lower part of SKN. The variables, which are used in deriving such signal, are depicted in Fig. 4. Considering this figure, the relation for the output signal can be derived:

$$K_F(t) = \int_0^T \left[K(t) - K_F(t)\right] / \left[C_F\, R_F\right] dt\,. \tag{39}$$

Differentiating both sides of (39), we get

$$dK_F(t)/dt = \left[K(t) - K_F(t)\right] / \left[C_F\, R_F\right]. \tag{40}$$

Eqn. 40 is solved as a homogeneous differential equation

$$dK_F(t)/dt = \left[-K_F(t)\right] / \left[C_F\, R_F\right], \tag{41}$$

which leads to

$$K_F(t) = C_1\, \exp\left[-t / \left(C_F\, R_F\right)\right], \tag{42}$$

Considering that $C_1$ is time-dependent:

$$\begin{aligned} &\frac{d\,C_1(t)}{dt} \exp\left(\frac{-t}{C_F\, R_F}\right) - \frac{C_1(t)}{C_F R_F} \exp\left(\frac{-t}{C_F\, R_F}\right) = \\ &= \frac{1}{C_F R_F} \left[K(t) - C_1(t) \exp\left(\frac{-t}{C_F\, R_F}\right)\right] \end{aligned} \,. \tag{43}$$

Eqn. 43 can be turned to

$$\frac{dC_1(t)}{dt} \exp\left(\frac{-t}{C_F R_F}\right) = \frac{K(t)}{C_F\, R_F}, \tag{44}$$

and $C_1(t)$ can be expressed as

$$C_1(t) = \int_0^T \frac{K(t)\, \exp\left[t / \left(C_F\, R_F\right)\right]}{C_F\, R_F} dt\,. \tag{45}$$

Eqn. 45 is solved using the substitution

$$b = U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11}\,. \tag{46}$$

Then, the relation (45) turns to

$$\begin{aligned} C_1(t) = \frac{R_{11}\, C_{int}\, R_{int}}{R_f\, C_F\, R_F} \exp\left[-\frac{R_{int}\, C_{int}\, R_{11}\, R'_{11}}{C_F\, R_F\, U_0\, R_f^2}\right] \cdot \\ \cdot \frac{1}{b} \int_0^T \exp\left[\frac{b}{C_F\, R_F\, U_0\, R_f^2}\right] db \end{aligned} \,, \tag{47}$$

In order to evaluate the integral in (47), the exponential function is approximated using [5]

$$e^x = \sum_{n=0}^{\infty} x^n / n! \cdot \tag{48}$$

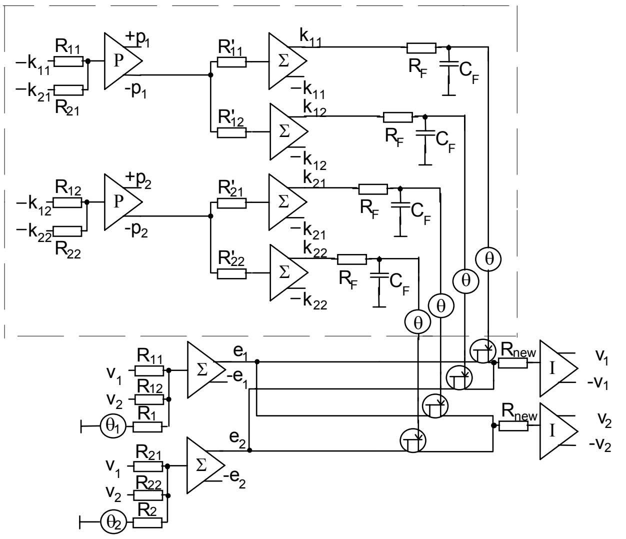

**Fig. 4** Analog realization of SKN with modified predictor

As a result, we get

$$C_1(t) = \ln\left| U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11} \right| \frac{R_{11}\, C_{int}\, R_{int}}{R_f\, C_F\, R_F} \cdot \exp\left[ -\frac{R_{int}\, C_{int}\, R_{11}\, R'_{11}}{C_F\, R_F\, U_0\, R_f^2} \right] + \sum_{n=1}^{\infty} \frac{\left( U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11} \right)^n}{n\; n! \left( C_F\, R_F\, U_0\, R_f^2 \right)^n} \frac{R_{11}\, C_{int}\, R_{int}}{R_f\, C_F\, R_F} \cdot \exp\left[ -\frac{R_{int}\, C_{int}\, R_{11}\, R'_{11}}{C_F\, R_F\, U_0\, R_f^2} \right] \tag{49}$$

Substituting (49) to (42) yields

$$K_F(t) = \ln\left| U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11} \right| \frac{R_{11}\, C_{int}\, R_{int}}{R_f\, C_F\, R_F} \cdot \exp\left[ -\frac{U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11}}{C_F\, R_F\, U_0\, R_f^2} \right] + \sum_{n=1}^{\infty} \frac{\left( U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11} \right)^n}{n\; n! \left( C_F\, R_F\, U_0\, R_f^2 \right)^n} \frac{R_{11}\, C_{int}\, R_{int}}{R_f\, C_F\, R_F} \cdot \exp\left[ -\frac{U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11}}{C_F\, R_F\, U_0\, R_f^2} \right] \tag{50}$$

The sub-circuit providing solution can be described by

$$e(t) = -\frac{R_f}{C_{int}\, R_{int}} \int_0^T \frac{e(t)\, K_F(t)}{R_{11}} dt - \frac{\theta_1\, R_f}{R_1}. \tag{51}$$

Differentiating both sides, we get

$$\frac{de(t)}{dt} = -\frac{e(t)\, R_f\, K_F(t)}{R_{11}\, C_{int}\, R_{int}}. \tag{52}$$

From this, $e(t)$ can be expressed as

$$e(t) = C_2 \exp\left[ \int -\frac{R_f\, K_F(t)}{R_{11}\, C_{int}\, R_{int}} dt \right]. \tag{53}$$

Here, $C_2$ is a constant, which can be determined from the assumption that the first term of (51) equals to zero at the beginning of the convergence process (it corresponds to the output of the integrator). Then, $e(0)$ equals to the second term of (51), and

$$C_2 = -\theta_1\, R_f \exp\left[ F(0) \right] / R_1 , \tag{54}$$

where

$$F(0) = \int \frac{R_f\, K_F(t)}{R_{11}\, C_{int}\, R_{int}} dt \Bigg|_{for\ t=0} , \tag{55}$$

and

$$e(t) = -\frac{\theta_1\, R_f}{R_1} \exp\left\{ F(0) - \int \frac{R_f\, K_F(t)}{R_{11}\, C_{int}\, R_{int}} dt \right\}. \tag{56}$$

Then, the solution signal can be expressed as

$$v(t) = \int \frac{e(t)\, K_F(t)}{C_{int}\, R_{int}} dt + C_3 , \tag{57}$$

which can be turned to

$$v(t) = -\frac{\theta_1\, R_{11}}{R_1} \exp\left[ F(0) - \int \frac{R_f\, K_F(t)}{R_{11}\, C_{int}\, R_{int}} dt \right] + C_3 . \tag{58}$$

Now, the time integral in this relation is solved. First, the first part of $K_F(t)$ is integrated:

$$F_1(t) = \int \frac{\ln\left| U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11} \right|}{C_F\, R_F} \cdot \exp\left[ -\frac{U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11}}{C_F\, R_F\, U_0\, R_f^2} \right] dt . \tag{59}$$

Employing the substitution (46), (59) is turned to

$$F_1(t) = \int \frac{\ln|b|}{C_F\, R_F\, U_0\, R_f^2} \exp\left[ -\frac{b}{C_F\, R_F\, U_0\, R_f^2} \right] db , \tag{60}$$

which can be solved according to [5] as

$$F_1(t) = -\ln\left| U_0\, R_f^2 t + R_{int}\, C_{int}\, R_{11}\, R'_{11} \right| \cdot \left[ \exp\left( -\frac{U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11}}{C_F\, R_F\, U_0\, R_f^2} \right) - 1 \right] + \sum_{n=1}^{\infty} \frac{\left( U_0\, R_f^2\, t + R_{int}\, C_{int}\, R_{11}\, R'_{11} \right)^n}{(-1)^n\; n\; n! \left( C_F\, R_F\, U_0\, R_f^2 \right)^n} . \tag{61}$$

If $t$ approaches infinity then the first term goes to positive infinity. Due to the negative sign in the argument of exponential function in (61), this make the $v(t)$ to approach the true solution of a task. The second term of (61) approaches zero (comparing this term and eqn. 48). In the case of exponential function with negative argument, the right side of (48) is multiplied by $(-1)^n$, which also occurs in (61). Such expression approaches zero for $t$ going to infinity.

The other part of the time integral from (58) is

$$F_2(t) = \int \frac{1}{C_F R_F} \sum_{n=1}^{\infty} \frac{\left(U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11}\right)^n}{n\, n!\left(C_F R_F U_0 R_f^2\right)^n} \cdot \cdot \exp\left[-\frac{U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11}}{C_F R_F U_0 R_f^2}\right] dt \tag{62}$$

Eqn. 62 is rearranged into the following form

$$F_2(t) = \sum_{n=1}^{\infty} \int \frac{\left(U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11}\right)^n}{C_F R_F\, n\, n!\left(C_F R_F U_0 R_f^2\right)^n} \cdot \cdot \exp\left[-\frac{U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11}}{C_F R_F U_0 R_f^2}\right] dt \cdot \tag{63}$$

Employing the substitution (46), (63) is turned to

$$F_2(t) = \sum_{n=1}^{\infty} \frac{\int b^n \exp\left[-b/\left(C_F R_F U_0 R_f^2\right)\right] db}{U_0 R_f^2 C_F R_F\, n n!\left(C_F R_F U_0 R_f^2\right)^n} \cdot \tag{64}$$

The integral can be evaluated according to [5]

$$F_2(t) = \sum_{n=1}^{\infty} \frac{1}{U_0 R_f^2 C_F R_F\, n n!\left(C_F R_F U_0 R_f^2\right)^n} \cdot \cdot \sum_{m=0}^{n} \left\{ \frac{n!\left(U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11}\right)^{n-m}}{(-1)^{m+1}(n-m)!\left(C_F R_F U_0 R_f^2\right)^{m+1}} \cdot \cdot \exp\left[-\frac{U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11}}{C_F R_F U_0 R_f^2}\right] \right\} . \tag{65}$$

Since (65) is similar to the second term of (61), we can show that $F_2(t)$ is approaching zero for $t$ going to infinity.

Now, the signal $v(t)$ can be expressed using (61), (65)

$$v(t) = -\left[\theta_1 R_{11}/R_1\right] \exp\left[F(0) - F_1(t) - F_2(t)\right] + C_3 . \tag{66}$$

Since the argument of the exponential function in (66) goes to zero for $t = 0$, we can show that

$$C_3 = \theta_1 R_{11}/R_1 . \tag{67}$$

Substituting (67) into (66)

$$v(t) = \left[\theta_1 R_{11}/R_1\right]\left\{1 - \exp\left[F(0) - F_1(t) - F_2(t)\right]\right\} . \tag{68}$$

In order to enable comparison of the original SKN and its improved version, we rearrange (68) to the form

$$\widetilde{v}(t) = \left[\theta_1 R_{11}/R_1\right] \left\{1 - \exp\left[F(0) - 2\exp(-x)\right] \cdot \cdot \left[U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11}\right]^{\exp(-x)-1}\right\} . \tag{69}$$

with

$$x = \frac{U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11}}{C_F R_F U_0 R_f^2} .$$

The term $(U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11})^{\exp(-x)-1}$ in (69) corresponds to $(U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11})^{-1}$ in (26). Both the terms are identical when $\exp(-x) = 0$, i.e. when $t$ approaches infinity. Nevertheless, convergence of the above term in (69) is slower than convergence of its counterpart in (26) due to the higher value of the power.

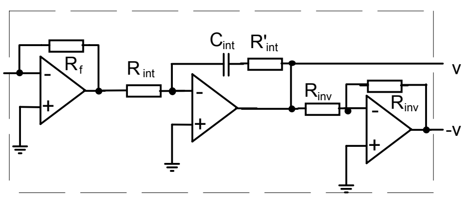


**Fig. 5** The modified integrator

Dealing with the function $\exp[F(0) - 2\exp(-x)]$, its value is higher than 1 and converges to 1 for time approaching infinity. Therefore, this term slows down the convergence.

Surprisingly, computer simulations show higher convergence rate of the improved SKN. This fact can be explained by the influence of real circuit components whereas the analysis is based on ideal components, computer simulations exploited more realistic models.

Dealing with eigenvalue ratio of the input autocorrelation matrix, its influence to the convergence properties is hardly being rigorously discussed. We can state only that term $(U_0 R_f^2 t + R_{int} C_{int} R_{11} R'_{11})^{\exp(-x)-1}$ in (69) plays a similar role as in the classical SKN because the exponential in the power varies from 0 to 1 during settling. Similar situation can be observed at the second term of (69).

## 6. SKN with modified integrator

Another modification of SKN employs an additional resistor in series with $C_{int}$ in the integrator sub-circuit (see Fig. 5). Influence of the resistor is investigated by deriving a formula for the error signal and for the solution one.

The modified circuit can be described by

$$e(t) = -\frac{R_f}{C_{int} R_{int}} \int_0^T \frac{e(t) K(t)}{R_{11}} dt - -\frac{e(t) K(t) R'_{int} R_f}{R_{int} R_{11}} - \frac{\theta_1 R_f}{R_1} , \tag{70}$$

where $K(t)$ is Kalman gain (17). Differentiating both sides,

$$\frac{de(t)}{e(t)} = \frac{\dfrac{R_f\,K(t)}{R_{11}\,C_{int}\,R_{int}} + \dfrac{dK(t)}{dt}\,\dfrac{R'_{int}\,R_f}{R_{int}\,R_{11}}}{1+\dfrac{K(t)\,R_{int}\,R_f}{R_{int}\,R_{11}}}\,dt\,. \tag{71}$$

Hence, $e(t)$ can be expressed as

$$e(t) = C_1\,\exp\left[-A(t)\right], \tag{72}$$

where

$$A(t) = \int\left[\frac{R_f\,K(t)}{R_{11}\,C_{int}\,R_{int}} + \frac{dK(t)}{dt}\,\frac{R'_{int}\,R_f}{R_{int}\,R_{11}}\right]\cdot \cdot\,\frac{R_{int}\,R_{11}}{R_{int}\,R_{11} + K(t)\,R'_{int}\,R_f}\,dt \tag{73}$$

and $C_1$ is a constant, which can be determined using initial conditions. Integrating (73), $b = K(t)$ is used as an integration variable:

$$A(t) = -\int\left[\frac{1}{b} - \frac{R'_{int}\,R_f}{R_{int}\,R_{11}}\right]\frac{R_{int}\,R_{11}}{R_{int}\,R_{11} + b\,R'_{int}\,R_f}\,db\,. \tag{74}$$

Introducing substitution

$$a = 1 + \frac{b\,R'_{int}\,R_f}{R_{int}\,R_{11}}\ , \tag{75}$$

eqn. 74 can be rewritten to the form:

$$A(t) = -\int\left[\frac{1}{a(a-1)} - \frac{1}{a}\right]da. \tag{76}$$

Eqn. 76 can be turned to

$$A(t) = \ln a - \int\left(\frac{1}{a-1} - \frac{1}{a}\right)da = = -\ln\left\{\frac{R_f\,R'_{int}\,K(t)}{R_{11}\,R_{int}}\left[\frac{R_{11}\,R_{int}}{R_{11}\,R_{int} + R_f\,R'_{\mathrm{int}}\,K(t)}\right]^2\right\}.$$

Finally, the error signal can be expressed as

$$e(t) = C_1\,\frac{R_f\,R'_{int}\,K(t)}{R_{11}\,R_{int}}\left[\frac{R_{11}\,R_{int}}{R_{11}\,R_{int} + R_f\,R'_{int}\,K(t)}\right]^2. \tag{78}$$

The solution signal can be obtained from the relation

$$e(t) = v(t)\,R_f/R_{11} - \theta_1\,R_f/R_1\,, \tag{79}$$

that can be derived from (18). Using (79), $v(t)$ is of a form

$$v(t) = \frac{R_{11}}{R_f}\left\{\frac{\theta_1\,R_f}{R_1} + \right. \left. + C_1\,\frac{R_f\,R'_{int}\,K(t)}{R_{11}\,R_{int}}\left[\frac{R_{11}\,R_{int}}{R_{11}\,R_{int} + R_f\,R'_{int}\,K(t)}\right]^2\right\}. \tag{80}$$

In order to evaluate $C_1$, we have to find $e(0)$. This can be obtained from the equation (derived from eqn. 79 for $t = 0$)

$$e(0) = -\frac{e(0)\,K(0)\,R'_{int}\,R_f}{R_{int}\,R_{11}} - \frac{\theta_1\,R_f}{R_1}\,, \tag{81}$$

Then, $e(0)$ can be expressed as

$$e(0) = -\frac{\theta_1\,R_f}{R_1}\,\frac{R_{11}\,R_{int}}{R_{11}\,R_{int} + R_f\,R'_{int}\,K(0)}\,. \tag{82}$$

Exploiting (78), the initial error $e(0)$ can be expressed as

$$e(0) = C_1\,\frac{R_f\,R'_{int}\,K(0)}{R_{11}\,R_{int}}\left[\frac{R_{11}\,R_{int}}{R_{11}\,R_{int} + R_f\,R'_{int}\,K(0)}\right]^2. \tag{83}$$

Therefore, we can express $C_1$ as

$$C_1 = -\frac{\theta_1\,R'_{11}\,R_{11}\,R_{int}}{U_0\,R_1\,R_f\,R'_{int}}\left(1 + \frac{R_f^2\,R'_{int}\,U_0}{R'_{11}\,R_{11}\,R_{int}}\right), \tag{84}$$

and the solution is of the form

$$v(t) = -\frac{\theta_1\,R_{11}}{R_1}\,\frac{\left[R_{int}\,C_{int}\,R_{11}\,R'_{11} + R_f^2\,R'_{int}\,U_0\,C_{int}\right]}{\left[U_0\,R_f^2\,t + R_{int}\,C_{int}\,R_{11}\,R'_{11} + R_f^2\,R'_{int}\,U_0\,C_{int}\right]^2}\cdot \cdot\left[U_0\,R_f^2\,t + R_{\mathrm{int}}\,C_{\mathrm{int}}\,R_{11}\,R'_{11}\right] + \theta_1\,R_{11}/R_1$$

The above relation can be rewritten into the matrix form:

$$\mathbf{v}(t) = \mathbf{A}^{-1}\,\mathbf{b} - \left\{\left[\frac{C_{int}\,R_{int}}{U_0\,\rho t}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)^{-1} + \frac{R'_{int}\,C_{int}}{t}\right]^{-1} + \mathbf{I}\right\}^{-1}\cdot \cdot\left\{\left[\frac{t}{R'_{int}\,C_{int}} + \frac{R_{int}}{R'_{int}\,U_0\,\rho}\left(\mathbf{A}\mathbf{A}^{\mathrm{T}}\right)^{-1}\right]^{-1} + \mathbf{I}\right\}^{-1}\mathbf{A}^{-1}\,\mathbf{b}\,. \tag{86}$$

or using previously introduced variables:

$$\mathbf{v}(t) = \mathbf{A}^{-1}\,\mathbf{b}' - \left[\frac{t\,\Lambda}{C_{int}\,R_{int}}\left(\Lambda\frac{R'_{int}}{R_{int}} + \frac{1}{U_0\,\rho}\right)^{-1} + \mathbf{I}\right]^{-1}\cdot \cdot\left[\frac{R'_{int}}{R_{int}}\left(\frac{t}{R'_{int}\,C_{int}} + \frac{\Lambda^{-1}}{U_0\,\rho}\right)^{-1} + \mathbf{I}\right]^{-1}\mathbf{A}^{-1}\,\mathbf{b}'\,. \tag{87}$$

Content of brackets on the first line corresponds to (38) in the case $R'_{int} = R_{int}$. The term in brackets on the second line does not contain time multiplied by $\Lambda$ (hence for $t$ being sufficiently high, influence eigenvalues is small). Unfortunately, this influence occurs at the beginning of the convergence process, which is disadvantage comparing to (38), but the convergence is better than classical SKN exhibits.

## 7. Combination of improvements

The above described improvements of the original SKN are combined in order to reach high convergence rate and small influence of eigenvalues. Due to the rather high complexity, the *combined* network has not been modeled mathematically, and its parameters have been verified by computer simulations only (Tab. 1).

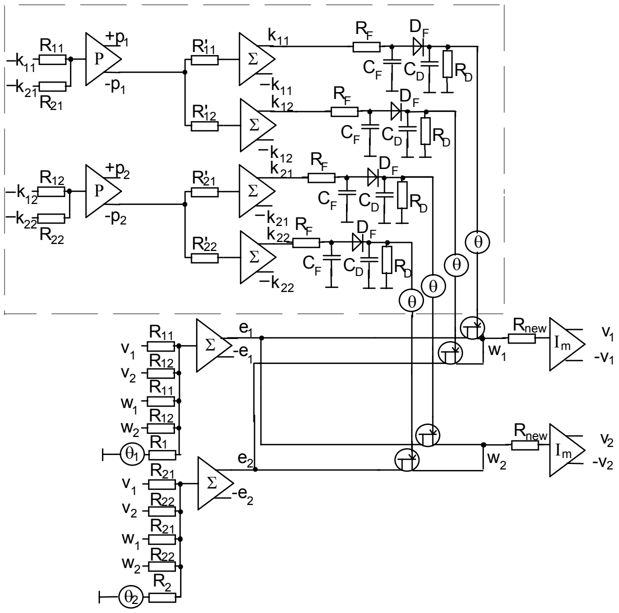

**Fig. 6** Analog realization of the improved SKN for solving set of simultaneous linear equations

Except of the analyzed improvements, even further modifications have been developed, whose functionality has been tested by computer simulations only:

1. The parallel combination of diodes $D_1$, $D_2$ and of a resistor $R_s$ has been used in an integrator (Fig. 7).
2. The capacitor $C'_{int}$ has been incorporated into the integrator (Fig. 7).

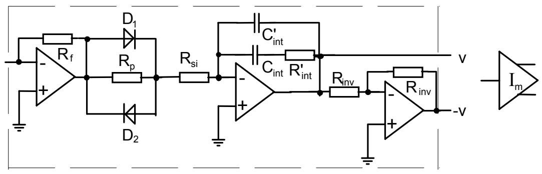

**Fig. 7** A modified integrator

The importance of the modifications of SKN, depicted in Fig. 6, is documented by computer simulations presented in Section 8: convergence properties of the modified net are enhanced such way that modified SKN can be used in such real time applications as adaptive antennas, e.g.

# 8. Computer simulations

In Tab. 1, convergence rate of the original SKN (Fig. 2) and modified one (Fig. 6) are compared. Both networks were set to exhibit the optimal convergence time for the following set of equations:

$$\begin{pmatrix} 50 & 1 \\ 1 & 0.2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{88}$$

with the eigenvalue ratio of the matrix $\mathbf{AA}^{\mathrm{T}}$

$$\lambda_2 / \lambda_1 = 70000 \ . \tag{90}$$

The convergence time of both the networks is given in the 3rd row of Tab. 1. Other results were obtained with the same setting of circuit parameters for these sets of equations:

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ , \ \lambda_2 / \lambda_1 = 7 \tag{91}$$

and

$$\begin{pmatrix} 3 & 1 \\ 1 & 0.2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ , \ \lambda_2 / \lambda_1 = 700 \ . \tag{92}$$

| | **modified SKN** | **Wang's net** |
|---|---|---|
| $\lambda_2 / \lambda_1$ | ***t*** **[μs]** | ***t*** **[μs]** |
| **7** | **1.23** | **85.79** |
| **700** | **1.05** | **463.16** |
| **70000** | **1.21** | **2953.10** |

**Tab. 1** Dependency of the convergence time on the eigenvalue ratio of the input signal matrix

# 9. Conclusion

The paper presents original improvements of the analog neural networks based on simplified Kalman filter. In sections 4 to 6, several improvements of the original SKN were presented and analyzed. In section 7, improvements were used for creating very fast, modified SKN, which exhibits very high convergence rate and very low dependence on the eigenvalue ratio. Properties of modified SKNs were verified using computer simulations.

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