Difference Equations with Forward and Backward Differences and Their Usage in Digital Signal Processor Algorithms

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Abstract. *In the paper the relation is given between linear difference equations with constant coefficients those obtained via the application of forward and backward differences. Relation is also established between input-output difference equations and state-space difference equations, which define the state of inner quantities of a discrete system. In conclusion, the state-space representation of a discrete system is given, which is suitable for implementing a discrete system in the microprocessor and digital signal processor. The resultant solution consists of the response to input signal and the response to non-zero initial conditions.*

Keywords

Forward and backward differences, difference equations, digital signal processor algorithms.

1. Introduction

Difference equations that represent algorithms for the digital signal processing can be expressed in several ways. In the field of digital signal processing, in particular in digital filters, the representation using the backward difference is mostly used [6], [7], [8]. In mathematics and in the field of informatics and automatic control, difference equations with forward differences are used [1], [2], [3], [4]. Difference equations can further be divided into space-state difference equations [2], [3], [4], [12], and [13] and input-output difference equations [1], [7], [8]. Each of these four types of difference equation uses a different method of programming algorithms. An incorrectly chosen type of algorithm can then often lead to errors when implementing the algorithm, in particular in the case of fixed-point digital signal processors. For a graphical representation of difference equations both block diagrams and signal flow graphs can be used [2, 3, 4, and 7]. In addition to the classical mathematical analysis [1] the Z-transform is frequently used [5, 6] to solve difference equations. In reverse examination of the transfer function implemented in the Z-transform, Mason's gain rule can be used for the calculation of graph transfer from the chosen input node to the output node [7]. The computer analysis of the realization structures of discrete systems is performed much the same as with the method of nodal voltages in analog circuits and systems, and is of fundamental significance in the analysis of the properties of discrete systems [10, 11]. To make full use of all the knowledge obtained so far by applying difference equations in the representation of discrete systems it is necessary to put the different notations of difference equations and their derived functions into a correct mutual relationship.

2. Difference Equations with Forward and Backward Differences

In mathematics the term difference equations refers to equations in which in addition to the argument and the sought function of this argument there are also their differences. The difference equation can be understood as the function:

$$y[n+1] = f\left(y[n]\right). \tag{1}$$

Function differences were first defined as approximations of function derivatives since for an n-th continuous derivative of function $f(x)$ at point x it holds:

$$f^{(n)}(x) = \lim_{\Delta x \to 0} \frac{\Delta^n f(x)}{\Delta x^n}. \tag{2}$$

If we consider a uniform (equidistant) distribution of the values of independent variable x, we can successively modify the function difference to the form:

$$\Delta f[n] = f[n+1] - f[n]. \tag{3}$$

This equation defines the so-called *forward difference.* In this way, the backward difference equation of the first order can also be defined:

$$\Delta_z f[n] = f[n] - f[n-1]. \tag{4}$$

The properties of continuous systems are mostly described using the differential equations. Similarly, the properties of discrete systems can be defined using the difference equations. Consider first a linear stationary discrete system of the s-th order whose properties can be represented by a linear difference equation of the s-th order with constant coefficients [1], [8]:

$$\begin{aligned} & B_s\,\Delta^s y[n]+B_{s-1}\,\Delta^{s-1}y[n]+B_{s-2}\,\Delta^{s-2}y[n]+ \\ & +B_{s-3}\,\Delta^{s-3}y[n]+\cdots+B_1\,\Delta y[n]+B_0\,y[n]= \\ & =A_s\,\Delta^s x[n]+A_{s-1}\,\Delta^{s-1}x[n]+A_{s-2}\,\Delta^{s-2}x[n]+ \\ & \cdots+A_1\,\Delta x[n]+A_0\,x[n]. \end{aligned} \tag{5}$$

The constants $B_0, B_1, \ldots, B_s$ and $A_0, A_1, \ldots, A_s$ are the difference equation coefficients, the discrete signal $x[n]$ represents the input signal of discrete system, and $y[n]$ is the output signal of discrete system. As can be seen, difference equation (5) describes the properties of a discrete system with one input and one output. If the discrete system has several inputs and several outputs, difference equations must be set up for each input and each output. It is then of advantage to use the matrix notation for a group of these difference equations. To obtain an analytical solution of equation (5) we need to know the initial conditions

$$\Delta^{s-1}y[0],\,\Delta^{s-2}y[0],\,\Delta^{s-3}y[0],\cdots,\Delta\,y[0],\,y[0]\;.$$

If we use the forward difference according to (3) to rewrite equation (5), then the general relation for s-th order difference can be used:

$$\begin{aligned} & \Delta^s f[n]=\Delta^{s-1}f[n+1]-\Delta^{s-1}f[n]= \\ & =\sum_{i=0}^{s}\,(-1)^s\,\frac{s!}{i!(s-i)!}\,f[n+s-i]. \end{aligned} \tag{6}$$

Using equation (6) we can obtain from rewritten equation (5) the following form:

$$\begin{aligned} & b_s\,y[n+s]+b_{s-1}\,y[n+s-1]+b_{s-2}\,y[n+s-2]+ \\ & +\cdots+b_1\,y[n+1]+b_0\,y[n]= \\ & =a_s\,x[n+s]+a_{s-1}\,x[n+s-1]+ \\ & +\cdots+a_1\,x[n+1]+a_0\,x[n]. \end{aligned} \tag{7}$$

This difference equation will be referred to as non-homogeneous linear difference equation of the s-th order with constant coefficients a_i, b_i, $i = 0, 1, ..., s–1, s$. This type of difference equation is sometimes referred to as recurrent equation of the s-th order, because we can write:

$$\begin{aligned} & y[n+s]=-\frac{b_{s-1}}{b_s}y[n+s-1]-\frac{b_{s-2}}{b_s}y[n+s-2]+ \\ & +\cdots-\frac{b_1}{b_s}y[n+1]-\frac{b_0}{b_s}y[n]+ \\ & +\frac{a_s}{b_s}x[n+s]+\frac{a_{s-1}}{b_s}x[n+s-1]+ \\ & +\cdots+\frac{a_1}{b_s}x[n+1]+\frac{a_0}{b_s}x[n]. \end{aligned} \tag{8}$$

The complete solution of non-homogeneous equation (7) consists of two parts:

$$y[n]=y_{\mathrm{h}}[n]+y_{\mathrm{p}}[n]. \tag{9}$$

The general solution $y_{\mathrm{h}}[n]$ of homogeneous difference equation will be obtained by solving equation (7) with zero right-hand side under non-zero initial conditions:

$$y[s-1],\,y[s-2],\cdots,\,y[1],\,y[0]. \tag{10}$$

The particular solution of $y_{\mathrm{p}}[n]$ depends on the form of the right-hand side of equation (7). The analytical solution is a complete solution but it is often very difficult. Analytical solution for different types of difference equations is described, for example, in [1]. For practical tasks of digital signal processing there are certain limitations, and for the solution the one-sided Z-transform is used [5], [6], [7]:

$$F(z)=Z\{f[n]\}=\sum_{n=0}^{\infty}\,f[n]\,z^{-n}\Leftrightarrow f[n]. \tag{11}$$

In the solution of difference equation (7) via the Z-transform in the form of (11) the following property of the transform is exploited:

$$\begin{aligned} & f[n+s]\Leftrightarrow Z\{f[n+s]\}= \\ & =z^s\left(F(z)-\sum_{i=0}^{s-1}\,f[i]\,z^{-i}\right). \end{aligned} \tag{12}$$

Using the Z-transform in the form of (11) and property (12) will yield equation (7) in the form:

$$\begin{aligned} & Y(z)=H(z)X(z)+ \\ & +\frac{1}{B_s(z)}\sum_{i=0}^{s-1}\{F_i(z)\,y[i]-G_i(z)\,x[i]\} \end{aligned} \tag{13}$$

where $Y(z)$ is the one-sided Z-transform of output signal $y[n]$, or $y[n] \Leftrightarrow Y(z)$, and $X(z)$ is the one-sided Z-transform of output signal $x[n]$, or $x[n] \Leftrightarrow X(z)$. The polynomials $F(z)$ and $G(z)$ are defined as follows:

$$\begin{aligned} & F_0(z)=b_s\,z^s+b_{s-1}\,z^{s-1}+\cdots+b_2\,z^2+b_1\,z\;, \\ & F_1(z)=b_s\,z^{s-1}+b_{s-1}\,z^{s-2}+\cdots+b_3\,z^2+b_2\,z\;, \\ & F_2(z)=b_s\,z^{s-2}+b_{s-1}\,z^{s-3}+\cdots+b_4\,z^2+b_3\,z\;, \\ & \vdots \\ & F_{s-3}(z)=b_s\,z^3+b_{s-1}\,z^{s-2}+b_{s-2}\,z\;, \\ & F_{s-2}(z)=b_s\,z^2+b_{s-1}\,z\;, \\ & F_{s-1}(z)=b_s\,z\;, \\ & G_0(z)=a_s\,z^s+a_{s-1}\,z^{s-1}+\cdots+a_2\,z^2+a_1\,z\;, \\ & G_1(z)=a_s\,z^{s-1}+a_{s-1}\,z^{s-2}+\cdots+a_3\,z^2+a_2\,z\,, \\ & G_2(z)=a_s\,z^{s-2}+a_{s-1}\,z^{s-3}+\cdots+a_4\,z^2+a_3\,z\;, \\ & \vdots \end{aligned} \tag{14}$$

$$\vdots$$
$$G_{s-3}(z) = a_s z^3 + a_{s-1} z^{s-2} + a_{s-2} z ,$$
$$G_{s-2}(z) = a_s z^2 + a_{s-1} z ,$$
$$G_{s-1}(z) = a_s z.$$

In [14] a comparison is given of the analytical solution and the solution using the Z-transform for 2nd-order non-homogeneous equation, which has the form:

$$y[n+2] - 1.25 y[n+1] + 0.78125 y[n] = \\ = x[n+2] - x[n].$$

The input signal considered is unity impulse $x[n] = \delta[n]$. The function $H(z)$ in equation (13) is called the system transfer function of discrete system and is defined as:

$$H(z) = \frac{A_s(z)}{B_s(z)} = \\ = \frac{a_s z^s + a_{s-1} z^{s-1} + a_{s-2} z^{s-2} + a_{s-3} z^{s-3} + \cdots}{b_s z^s + b_{s-1} z^{s-1} + b_{s-2} z^{s-2} + b_{s-3} z^{s-3} + \cdots} \\ \frac{\cdots + a_2 z^2 + a_1 z + a_0}{\cdots + b_2 z^2 + b_1 z + b_0} . \tag{15}$$

Linear stationary (time-invariant) discrete dynamic systems also embrace the IIR and FIR digital filters. In their design the transfer function $H(z)$ is mostly started from, which is written in a form other than (15):

$$H(z) = \frac{c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \cdots}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3} + \cdots} \\ \frac{\cdots + c_{s-2} z^{-s+2} + c_{s-1} z^{-s+1} + c_s z^{-s}}{\cdots + d_{s-2} z^{-s+2} + d_{s-1} z^{-s+1} + d_s z^{-s}} = \frac{Y(z)}{X(z)}. \tag{16}$$

If we compare equations (15) and (16), we can see the connection:

$$c_0 = \frac{a_s}{b_s}, c_1 = \frac{a_{s-1}}{b_s}, c_2 = \frac{a_{s-2}}{b_s}, \cdots , \\ c_{s-1} = \frac{a_1}{b_s}, c_s = \frac{a_0}{b_s} , \\ d_1 = \frac{b_{s-1}}{b_s}, d_2 = \frac{b_{s-2}}{b_s}, d_3 = \frac{b_{s-3}}{b_s}, \cdots , \\ d_{s-1} = \frac{b_1}{b_s}, d_s = \frac{b_0}{b_s} . \tag{17}$$

Using different methods of digital filter design (bilinear transformation, the Remez algorithm, etc.) the coefficients a_i, b_i, $i = 0, 1, 2, \ldots , s$ (or c_i, d_i) are sought such that the requirements, for example, of the tolerance diagram for the width and placement of pass, stop and transition frequency bands of digital filter are satisfied [7], [8]. Now we are interested in what form of difference equation corresponds to a procedure that starts directly from transfer function (16). We calculate the difference equation from the equation in the Z-transform, which forms only a part of equation (13):

$$Y(z) = H(z) X(z) . \tag{18}$$

After substituting transfer function (16) into equation (18), using the inverse Z-transform and rewriting we obtain the difference equation:

$$y[n] + d_1 y[n-1] + d_2 y[n-2] + \cdots \\ + d_{s-2} y[n-s+2] + d_{s-1} y[n-s+1] + d_s y[n-s] = \\ = c_0 x[n] + c_1 x[n-1] + c_2 x[n-2] + \cdots \\ + c_{s-2} x[n-s+2] + c_{s-1} x[n-s+1] + c_s x[n-s] . \tag{19}$$

or after substituting coefficients a_i and b_i:

$$b_s y[n] + b_{s-1} y[n-1] + b_{s-2} y[n-2] + \cdots \\ + b_2 y[n-s+2] + b_1 y[n-s+1] + b_0 y[n-s] = \\ a_s x[n] + a_{s-1} x[n-1] + a_{s-2} x[n-2] + \cdots \\ + a_2 x[n-s+2] + a_1 x[n-s+1] + a_0 x[n-s] . \tag{20}$$

The recurrent equation can again be obtained via rewriting:

$$y[n] = -\frac{b_{s-1}}{b_s} y[n-1] - \frac{b_{s-2}}{b_s} y[n-2] - \cdots \\ -\frac{b_2}{b_s} y[n-s+2] - \frac{b_1}{b_s} y[n-s+1] - \frac{b_0}{b_s} y[n-s] + \\ + \frac{a_s}{b_s} x[n] + \frac{a_{s-1}}{b_s} x[n-1] + \frac{a_{s-2}}{b_s} x[n-2] + \cdots \\ + \frac{a_2}{b_s} x[n-s+2] + \frac{a_1}{b_s} x[n-s+1] + \frac{a_0}{b_s} x[n-s] . \tag{21}$$

The complete solution of non-homogeneous equation in the form of (20) or (21) will be obtained on the assumption that we know the initial conditions:

$$y[-1], y[-2], \cdots , y[-s+1], y[-s] . \tag{22}$$

As can be seen, we obtain the form of linear difference equations with backward differences (4), which is commonly used in digital filter theory.

Conclusion No. 1: If we start from a linear non-homogeneous difference equation with constant coefficients in the form of (5), then after substituting forward and backward differences according to (3) or (4) we obtain equivalent difference equations in the form of (7) and (8) or (20) and (21). We can see that we are concerned here with an equivalent form if we apply the substitution $n = m + s$ to equation (20) or (21). It should be noted that this equivalence only concerns linear difference equations with constant coefficients. This equivalence does not hold for nonlinear difference equations or linear difference equations with varying coefficients.

3. State-Space Difference Equations and Input-Output Difference Equations

Non-homogeneous difference equations in the form of (5), (7) or (8) only express the relation between input signal $x[n]$ and output signal $y[n]$ of a discrete system (or between the input and the output signals for a system with several inputs and outputs, in which case a group of difference equations is used). The form of input-output difference equation is not very suitable for writing algorithms because it is difficult to determine the value of initial conditions. As can be seen from conditions (10) and (22), we must determine s initial values of output signal $y[n]$, which are distributed in time and thus influence the rate of processing the input signal by the discrete system. Of greater advantage would be the requirement of initial values defined for one instant of time, which appears in the solution of difference equation (5). In mathematics, of course, it is not customary to do an analysis of difference equations in the form of (5); in most cases, difference equations in the form of (7) or (8) are solved. It would be more convenient if we introduced a model of the discrete system with inner variables. This model is then referred to as state-space model because it will give the values of inner state-space variables at one instant of time. Let us show a very simple method of deriving the inner state-space description. We will successively put together differently shifted input and output signals:

$$\begin{aligned}
&v_1[n] = b_s\, y[n] - a_s\, x[n],\\
&v_2[n] = b_{s-1}\, y[n] + b_s\, y[n+1] - a_{s-1}\, x[n] - a_s\, x[n+1],\\
&v_3[n] = b_{s-2}\, y[n] + b_{s-1}\, y[n+1] + b_s\, y[n+2] -\\
&- a_{s-2}\, x[n] - a_{s-1}\, x[n+1] - a_s\, x[n+2],\\
&\vdots\\
&v_s[n] = b_1\, y[n] + b_2\, y[n+1] + b_3\, y[n+2] + \cdots\\
&+ b_{s-1}\, y[n+s-2] + b_s\, y[n+s-1] -\\
&- a_1\, x[n] - a_2\, x[n+1] - a_3\, x[n+2] - \cdots\\
&- a_{s-1}\, x[n+s-2] - a_s\, x[n+s-1].
\end{aligned} \tag{23}$$

We have obtained a group of new state-space variables $v_i[n]$, $i = 1, 2, 3, \ldots, s$. Formally we can also write a difference equation that is in agreement with equation (7):

$$\begin{aligned}
&v_{s+1}[n] = b_0\, y[n] + b_1\, y[n+1] + b_2\, y[n+2] + \cdots\\
&+ b_{s-1}\, y[n+s-1] + b_s\, y[n+s]\\
&- a_0\, x[n] - a_1\, x[n+1] - a_2\, x[n+2] + \cdots\\
&- a_{s-1}\, x[n+s-1] - a_s\, x[n+s] = 0\,.
\end{aligned}$$

From the first equation in (23) we will calculate the output signal:

$$y[n] = \frac{1}{b_s} v_1[n] + \frac{a_s}{b_s} x[n] \tag{24}$$

and successively substitute this equation in other equations of (23). After rewriting we obtain the following state-space difference equations:

$$\begin{aligned}
&v_1[n+1] = v_2[n] - \frac{b_{s-1}}{b_s} v_1[n] + \left(a_{s-1} - b_{s-1}\frac{a_s}{b_s}\right) x[n],\\
&v_2[n+1] = v_3[n] - \frac{b_{s-2}}{b_s} v_1[n] + \left(a_{s-2} - b_{s-2}\frac{a_s}{b_s}\right) x[n],\\
&\vdots\\
&v_{s-1}[n+1] = v_s[n] - \frac{b_1}{b_s} v_1[n] + \left(a_1 - b_1\frac{a_s}{b_s}\right) x[n],\\
&v_s[n+1] = -\frac{b_0}{b_s} v_1[n] + \left(a_0 - b_0\frac{a_s}{b_s}\right) x[n].
\end{aligned} \tag{25}$$

Writing in matrix notation we obtain:

$$\mathbf{v}[n+1] = \mathbf{A}\,\mathbf{v}[n] + \mathbf{B}\,x[n], \tag{26}$$

where the vectors are defined as follows:

$$\mathbf{v}[n+1] = \left[v_s[n+1]\; v_{s-1}[n+1] \cdots v_2[n+1]\; v_1[n+1]\right]^{\mathrm{T}}$$

and

$$\mathbf{v}[n] = \left[v_s[n] \quad v_{s-1}[n] \quad \cdots \quad v_2[n] \quad v_1[n]\right]^{\mathrm{T}}\,.$$

The symbol $^{\mathrm{T}}$ expresses the vector transposition operation. Matrices **A** and **B** are in the form:

$$\mathbf{A} = \begin{bmatrix}
0\,0\,0 & \cdots & 0\,0 & -\dfrac{b_0}{b_s}\\
1\,0\,0 & \cdots & 0\,0 & -\dfrac{b_1}{b_s}\\
0\,1\,0 & \cdots & 0\,0 & -\dfrac{b_2}{b_s}\\
\vdots & & & \\
0\,0\,0 & \cdots & 1\,0 & -\dfrac{b_{s-2}}{b_s}\\
0\,0\,0 & \cdots & 0\,1 & -\dfrac{b_{s-1}}{b_s}
\end{bmatrix},\quad
\mathbf{B} = \begin{bmatrix}
a_0 - b_0\dfrac{a_s}{b_s}\\
a_1 - b_1\dfrac{a_s}{b_s}\\
a_2 - b_2\dfrac{a_s}{b_s}\\
\vdots\\
a_{s-2} - b_{s-2}\dfrac{a_s}{b_s}\\
a_{s-1} - b_{s-1}\dfrac{a_s}{b_s}
\end{bmatrix}.$$

Matrix notation for a group of state-space difference equations has the advantage that it can represent in a clear way not only a discrete system with one input and one output but also systems with many inputs and outputs. In that case signals $x[n]$ and $y[n]$ will be replaced by vectors. This manner of notation of difference equations is primarily employed in the field of automatic measurement and control. Similarly, for output signal $y[n]$ we obtain the matrix equation:

$$y[n] = \mathbf{C}\,\mathbf{v}[n] + \mathbf{D}\,x[n]. \tag{28}$$

In our case, matrices **C** and **D** are in the form:

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \dfrac{1}{b_s} \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} \dfrac{a_s}{b_s} \end{bmatrix}. \tag{29}$$

Conclusion No. 2: State-space description of difference equations provides a more advantageous description of the properties of a linear time-invariant discrete system with several inputs and several outputs. Initial conditions are defined by the state-space vector $\mathbf{v}[n]$ for all state-space quantities at one instant of time; for example, for $n = 0$ the initial conditions are defined by vector

$$\mathbf{v}[0] = \begin{bmatrix} v_s[0] & v_{s-1}[0] & \cdots & v_2[0] & v_1[0] \end{bmatrix}^{\mathrm{T}}.$$

To follow the dynamic change in the discrete system properties, matrix equation (26) is used, which determines the values of state quantities for the subsequent instant of time. Based on the knowledge of state-space vector $\mathbf{v}[n]$ the values of output signal or signals can be determined at any instant of time.

4. State-Space Difference Equations for Implementation on Digital Signal Processor

A limiting factor of implementing algorithms for digital signal processing in fixed-point digital signal processors is the representation of positive and negative numbers in data arithmetic-logic units (DALU) of digital signal processors. For the representation of binary numbers the two's complement in fraction representation is used. Its dynamic range of numbers is limited to the interval between the highest positive number $1 - 2^{-b}$ and the lowest negative number -1. The symbol b denotes the number of bits of the binary number. Although accumulators in the DALUs of digital signal processors have additional bits that increase this dynamic range in partial calculations, the algorithms need to be pre-modified lest the range is exceeded. The respective modification depends on the architecture of digital signal processor and on the algorithm structure. In the implementation of the properties of discrete systems, state-space difference equations are used that also start from the form of (23) but the subsequent modification is performed in a different way. In state-space equations no substitution is performed for output signal $y[n]$, which is obtained from equation (24). Eventually we obtain the following system of s first-order state-space difference equations:

$$\begin{aligned} v_1[n+1] &= a_{s-1}\, x[n] - b_{s-1}\, y[n] + v_2[n], \\ v_2[n+1] &= a_{s-2}\, x[n] - b_{s-2}\, y[n] + v_3[n], \\ v_3[n+1] &= a_{s-3}\, x[n] - b_{s-3}\, y[n] + v_4[n], \\ &\vdots \\ v_{s-1}[n+1] &= a_1\, x[n] - b_1\, y[n] + v_s[n], \\ v_s[n+1] &= a_0\, x[n] - b_0\, y[n]. \end{aligned} \tag{30}$$

The output equation remains the same as in the preceding case:

$$y[n] = \frac{1}{b_s}\, v_1[n] + \frac{a_s}{b_s}\, x[n]. \tag{31}$$

To solve the system of difference equations we will use the one-sided Z-transform. For individual Z-transforms of state-space variables in equations (30) we obtain:

$$\begin{aligned} V_1(z) &= a_{s-1}\, X(z) z^{-1} - b_{s-1}\, Y(z) z^{-1} + V_2(z) z^{-1} + v_1[0], \\ V_2(z) &= a_{s-2}\, X(z) z^{-1} - b_{s-2}\, Y(z) z^{-1} + V_3(z) z^{-1} + v_2[0], \\ V_3(z) &= a_{s-3}\, X(z) z^{-1} - b_{s-3}\, Y(z) z^{-1} + V_4(z) z^{-1} + v_3[0], \\ &\vdots \\ V_{s-1}(z) &= a_1\, X(z) z^{-1} - b_1\, Y(z) z^{-1} + V_s(z) z^{-1} + v_{s-1}[0], \\ V_s(z) &= a_0\, X(z) z^{-1} - b_0\, Y(z) z^{-1} + v_s[0]. \end{aligned} \tag{32}$$

The one-sided Z-transform of the equation is:

$$b_s\, Y(z) = V_1(z) + a_s\, X(z) \ . \tag{33}$$

Equation (32) is successively substituted into equation (33) and after rewriting we obtain:

$$Y(z) = H(z) X(z) + \sum_{i=1}^{s} H_i(z) v_i[0]. \tag{34}$$

The system transfer function is the same as in the equation (15):

$$\begin{aligned} H(z) &= \frac{A_s(z)}{B_s(z)} = \\ &\frac{a_s z^s + a_{s-1} z^{s-1} + a_{s-2} z^{s-2} + a_{s-3} z^{s-3} + \cdots}{b_s z^s + b_{s-1} z^{s-1} + b_{s-2} z^{s-2} + b_{s-3} z^{s-3} + \cdots} \\ &\frac{\cdots + a_2 z^2 + a_1 z + a_0}{\cdots + b_2 z^2 + b_1 z + b_0}. \end{aligned} \tag{35}$$

The remaining transfer functions in equation (34) have the form:

$$\begin{aligned} &H_1(z) = \frac{z^s}{B_s(z)},\ H_2(z) = \frac{z^{s-1}}{B_s(z)},\ H_3(z) = \frac{z^{s-2}}{B_s(z)}, \\ &\cdots\ H_{s-1}(z) = \frac{z^2}{B_s(z)},\ H_s(z) = \frac{z}{B_s(z)}\ . \end{aligned} \tag{36}$$

If using the inverse Z-transform we transform equation (34) into the time domain, we obtain the resultant solution of state-space difference equations (30) and (31) in the form:

$$y[n] = h[n] * x[n] + \sum_{i=1}^{s} h_i[n]\, v_i[0]. \tag{37}$$

Sequence $h[n]$ is the impulse response of discrete system, which we obtain using the inverse Z-transform of transfer function:

$$h[n]=\frac{1}{2\pi \mathrm{j}}\oint_{C} H(z)\, z^{n-1}\, \mathrm{d}z \,. \tag{38}$$

Similarly, impulse responses $h_i[n]$ can be determined from impulse functions $H_i[z]$. The symbol * denotes the operation of discrete convolution, which has the form:

$$h[n]*x[n]=\sum_{m=-\infty}^{\infty} h[m]\, x[n-m]. \tag{39}$$

Conclusion No. 3: First-order state-space difference equations in the form of (30) have advantageous properties for implementing discrete systems in the digital signal processor or in another type of microprocessor. The initial state of discrete system is defined by the values of state quantities $v_i[0]$. The response of the system to initial conditions is given by the sum of impulse responses $h_i[n]$ of identical form, which are only differently shifted in time. The form of impulse responses is defined by the nominator of system transfer function $H(z)$.

5. Example

On the example of a type IIR digital filter in the form of a three-band rejection filter we will show how important it is to select a correct realization algorithm that matches the given architecture of digital signal processor. Let us choose the currently most powerful fixed-point digital signal processor TMS320C6416 (Texas Instruments). Its architecture is of the VLIW type and the computation power of the digital signal processor is over 8000 MIPS. The amplitude frequency response of the rejection filter can be seen in Fig. 1.

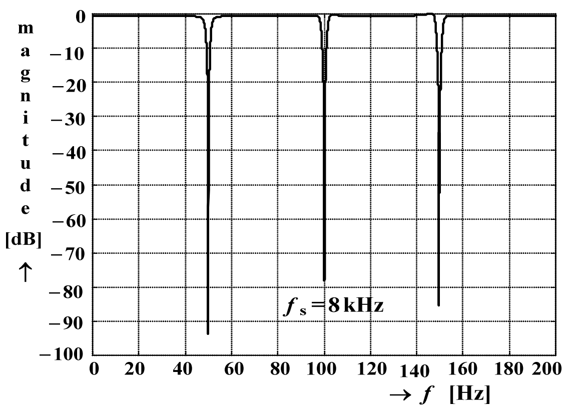

Fig. 1. Amplitude frequency response of 12th-order rejection filter.

Since the sampling frequency f_s is comparatively high in comparison with the centre frequencies of individual rejection filters, this digital filter is highly sensitive to the quantization of coefficients and intermediate results of arithmetic operations. Chosen for the realization was the classical structure of the series connection of 6 partial sections of 2nd order, which can be seen in Fig. 2.

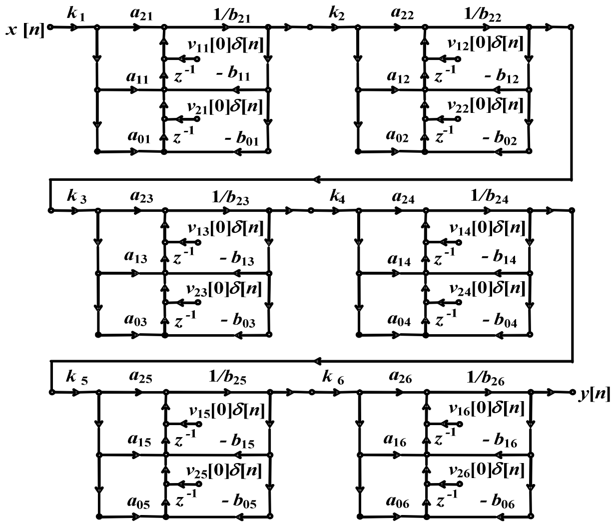

Fig. 2. Series connection of 6 partial sections of 2nd order for the realization of rejection filter.

The incorrect selection itself of the structure of partial section can make the algorithm non-functional. E.g., if we choose the 2nd canonic structure (Direct Form [7]) as the structure of partial sections, this filter cannot be implemented at all, since the values of internal state-space variables $v_{1i}[n]$ and $v_{2i}[n]$, i = 1, 2, 3, 4, 5, and 6, will exceed the permissible imaging range of numbers in two's complement. As an example, see the waveform of state-space variable $v_{1i}[n]$ of partial section No. 1 for the 1st canonic structure (Fig. 3) and the 2nd canonic structure (Fig. 4). Similar waveforms can be determined for the remaining state-space variables of further sections. Using the 2nd canonic structure we would have to choose a floating-point digital signal processor, which might be slower and more expensive for a given application. The description of the whole simulation procedure can be found in [9].

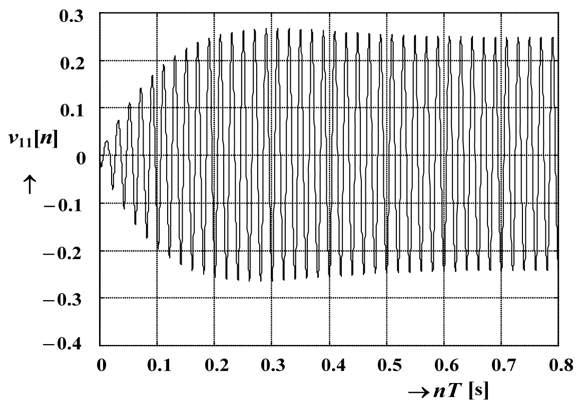

Fig. 3. State-space variable $v_{11}[n]$ of the first partial section of 2nd order, which is realized in the 1st canonic form.

In addition to selecting the optimum structure and coping with the effect of quantizing, etc., it is also important on how the given structure depends on the setting of non-zero initial conditions. This involves, for example, the form of transfer functions $H_i(z)$ in equation (34), which has been calculated for the 1st canonic structure of type IIR digital filters. Let the input signal be formed by the sum of 4 harmonic components of 50 Hz, 100 Hz, 150 Hz and 200 Hz frequencies in the form:

$$x[n] = 0.245\cos(2\pi\, 50\, n/f_s) + 0.245\cos(2\pi\, 100\, n/f_s) + \\ + 0.245\cos(2\pi\, 150\, n/f_s) + 0.245\cos(2\pi\, 200\, n/f_s)\ , \\ f_s = 8\ \text{kHz}\ . \qquad (40)$$

In Fig. 5 we can see the waveform and DFT spectrum, which can be displayed directly in the Code Composer Studio development environment, for filtering the input signal (40) in a form when all initial conditions are set to zero. The spectrum only contains the 200 Hz, harmonic component because the other components have been filtered by the rejection filter.

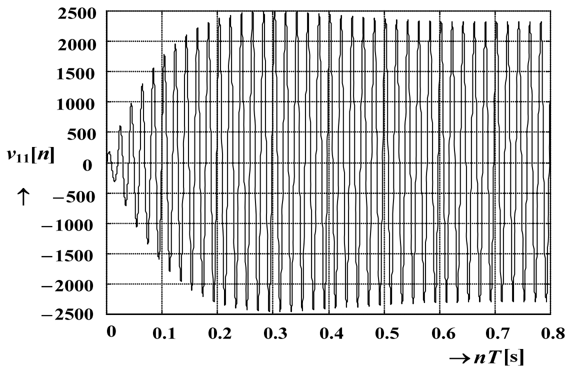

Fig. 4. State-space variable $v_{11}[n]$ of the first partial section of 2nd order, which is realized in the 2nd canonic form.

For the same filter and the same input signal we will now change all the initial conditions to a non-zero value (0.1). In Fig. 6 we can see that saturation will occur, i.e. the number range of the representation of numbers in two's complement will be exceeded, which will result in non- linear operations being inserted. Due to saturation the spectrum will spread and contain several components instead of one component.

This undesirable situation must be solved in several possible ways such that saturation does not occur. It is therefore important that prior to implementing the given algorithm in a digital signal processor a mathematical analysis should be performed such that the algorithm is best adapted to the architecture of the digital signal processor used.

6. Summary

Hybrid microcontrollers and fixed-point digital signal processors are frequently used in a great number of applications. These are in particular applications in telecommunications (modems, mobile phones, speech encoding, subchannel coding, etc.), in consumer electronics (type MPEG3 recorders, multimedia games, digital cameras, etc.), in automotive industry (various types of drive control), in computer technology (disk drive management), and the like. It is therefore important to seek optimum structures of realizing algorithms for digital signal processing. Linear stationary discrete systems are most frequently described by means of linear difference equations with constant coefficients. Best suited for implementation in fixed-point digital signal processors are state-space difference equations in the so-called canonic form, which are defined by equations (30). They have a homogeneous computation structure so that their application is advantageous for both digital signal processor with Harvard architecture and digital signal processors with type VLIW architecture, which use a high degree of parallelism in the processing.

In a similar way, other currently used structures of discrete systems can be analyzed, such as lattice, linear prediction, cepstral or continued-fraction-expansion structures, which are usually also described by systems of difference equations.

Acknowledgements

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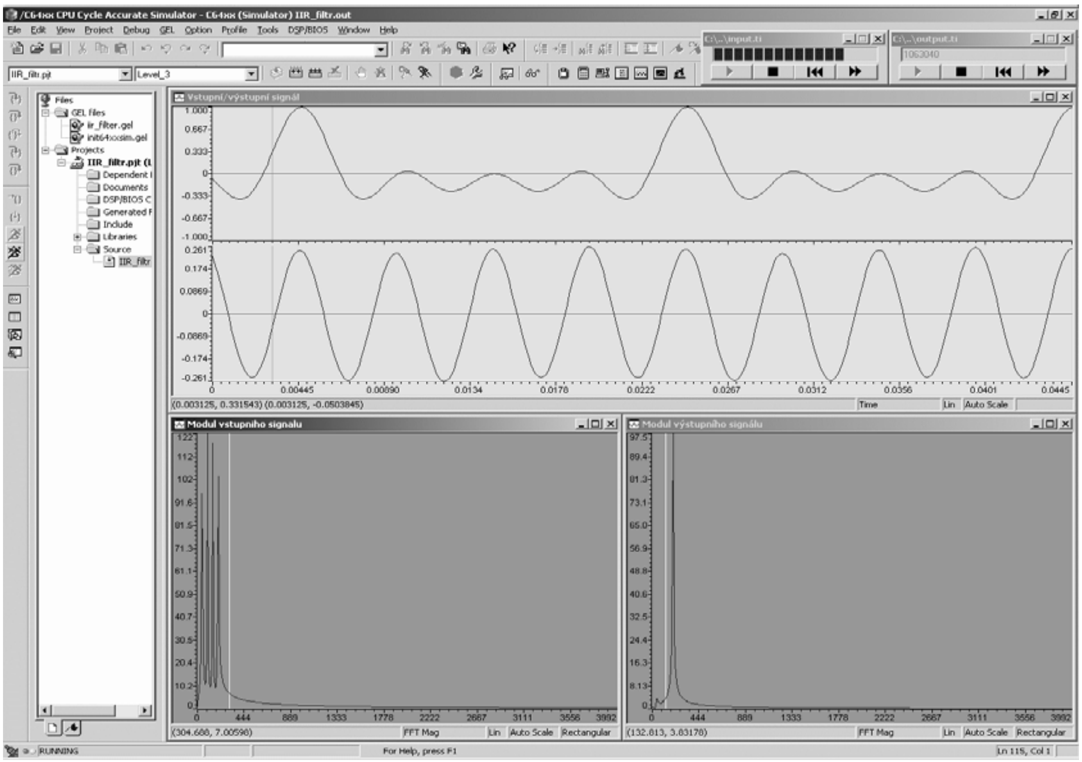

Fig. 5. Waveform and spectrum of input and output signals when filtering by rejection filter for zero initial conditions obtained by simulation on a TMS320C6416 digital signal processor in the Code Composer Studio development environment (Texas Instruments).

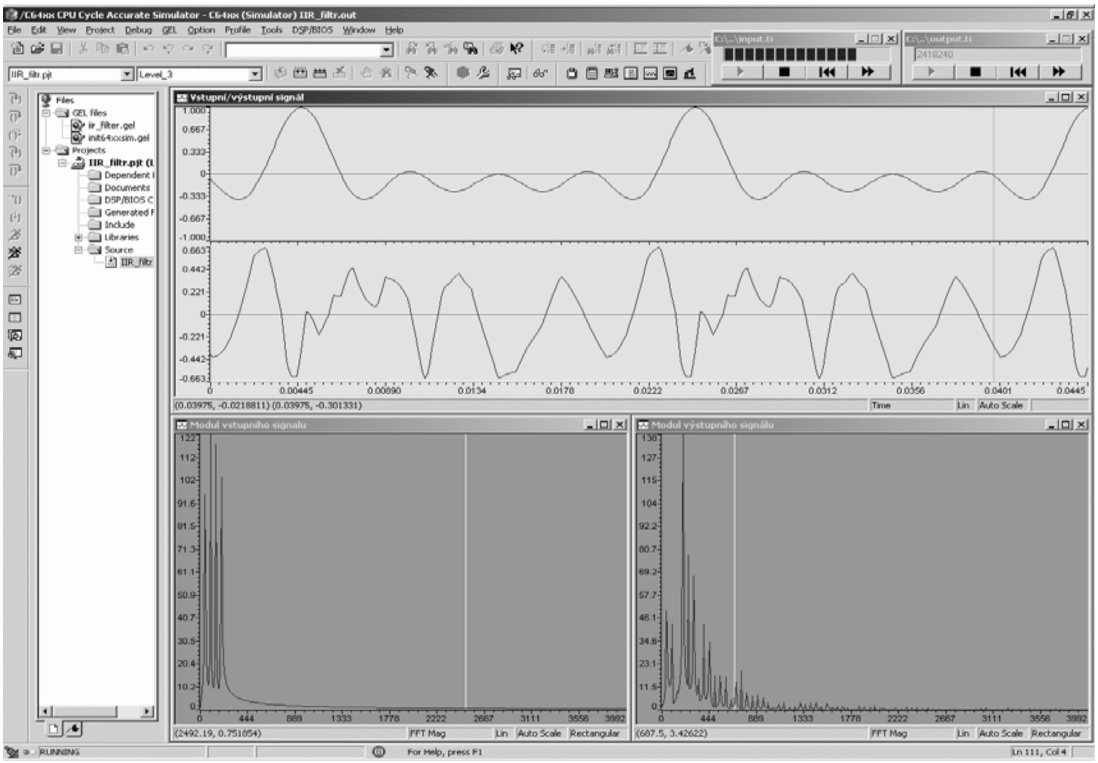

Fig. 6. Waveform and spectrum of input and output signals from Fig. 5 after all initial conditions have been set to non-zero.