The Conformal Correction
in the Strip-Centered Coaxial Line Analysis

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Abstract. *The presented work reports on the novel analysis of the strip-centered coaxial line. The conformal mapping method is used.*

For a narrow central strip a very easy conformal mapping is described that maps this structure onto the rectangular model line, but great distortion for wide strip inhibits application of this mapping in all the extent. The conformal mapping method allows to correct this fault very smartly by extension in auxiliary plane.

Two different numerical models are also mentioned as comparative solutions of strip-centered coaxial line analysis. One of them is based on numerical evaluating of the Schwarz-Christoffel integral, the other (in two modifications) is based on boundary element method (BEM).

Keywords

Strip-centered coaxial line, characteristic impedance, conformal mapping, conformal correction, numerical solution of Schwarz-Christoffel integral, BEM.

1. Introduction

A great boom of numerical method does not decrease the main advantage of the analytical methods - one mathematical function describes physical model. While in case of some numerical method we have to start whole analysis every time we change any parameter, analytical solution gives us result as quickly as we can enumerate one mathematical function. Also the influence of each particular parameter is clear from analytical solution, but not from every set of numerical results.

A lot of different electronic equipments have to work together. Unfortunately the power levels among them are over 200 dB very often. Coaxial structures are widely used because of their good shielding effect, which suppress the fields around strong distortion sources (e.g. transmitting antenna feeder) and protect sensitive parts of receivers, measurement inputs of the equipments etc. [1], [2].

Whereas coaxial line (two concentric cylindrical electrodes) is widely known, strip-centered coaxial line (SCCL, normalized on Fig. 1.) is mentioned rarely (founded only in very special literature like [3]).

The SCCL structure also offers a very attractive occasion of matching to microstrip line, coplanar waveguide, etc.
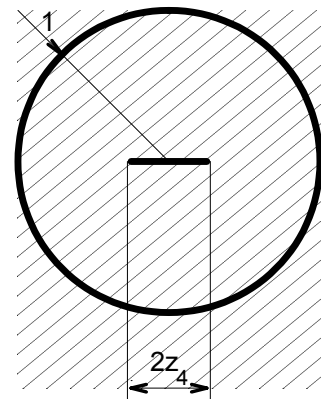

Fig. 1. Normalized strip-centered coaxial line.

2. Analysis

The basic assumption of the analysis is homogeneity of dielectric across the SCCL structure.

The arrangement of SCCL is symmetrical in two perpendicular planes. That convenient fact allows analyzing only one quarter of this structure. The open cuts are simply substituted by perfect magnetic walls.

2.1 Conformal Mapping

Let us assume the existence of the generalized triangle ($w_1w_3w_4$ - see Fig. 2b) in the plane w with the height $\pi/2$ and with one vertex (w_1) in infinity and let the origin of the Cartesian coordinates merge with the left bottom vertex (w_4). There is a transform $z \rightarrow w$, which maps the 1st quadrant of the plane z to this generalized triangle $w_1w_3w_4$ on the condition that the abscissa z_3z_4 maps onto the abscissa w_3w_4. The special version of Schwarz-Christoffel mapping (mentioned in [4]) is applied

$$\frac{dw}{dz^2} = \left(z^2\right)^{\frac{1}{2}}\left(z^2 - z_4^2\right)^{\frac{1}{2}}. \tag{1}$$

All the symbols are in agreement with Fig. 1. and Fig. 2. The integral of this expression takes us the transform formula

$$w = \operatorname{arccosh}(z/z_4).$$ (2)

What happens with the surrounding electrode (arc $z_2 z_5$)? This arc maps itself onto a curve $w_2 w_5$. In case of very small z_4 this curve is very close to abscissa orthogonal to x-axis. This could be a solution of the task, the characteristic impedance of the SCCL is proportional to the position of the point w_5 (in (8) w_5 instead of w_5^d) but for other z_4, greater than 0.2 (approx.), the difference between the ideal abscissa and our curve is not acceptable unfortunately. The correction has to be applied and the conformal method offers a very attractive way.
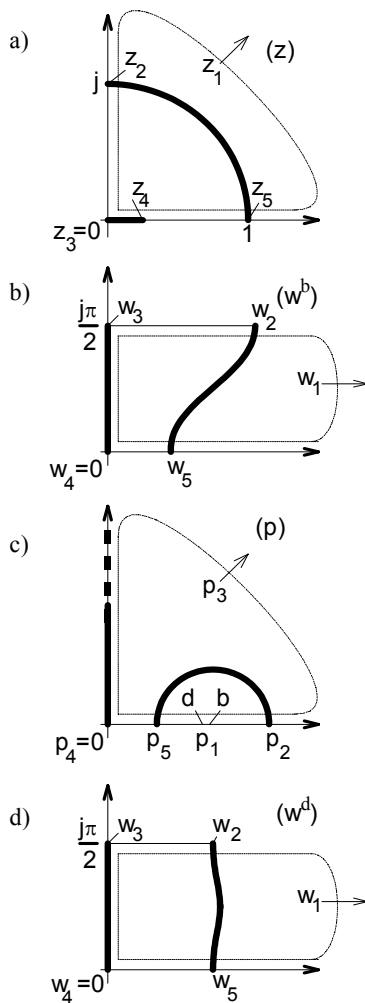

Fig. 2. Conformal mapping and correction.

2.2 Conformal Correction

The idea of the conformal correction (CC) is to transform our plane w to an auxiliary plane p and return back, but "little bit" modified.

For the sake of clarity, we rename our plane w (Fig. 2b) to w^b and the corrected plane w (Fig. 2d) to w^d. In

the plane p the mark b relates to the plane p before adjustment and the mark d after it.

In Svačina's work [4] there is the identical shape to our shape in the plane w^b. There it is used as the conformal map of the circular wire over a flat plane (approximation of a narrow microstrip). Because we interchange the cause and the result and transform the shape on the plane w^b to a "wire-over-plate" line, we use an inverse transform

$$p = \tanh w^b.$$ (3)

The infinity point $w_1^b \to \infty$ maps itself onto the point p_1^b. Now let us assume the arc $p_2 p_5$ is circular (it is not, in fact, but the difference is marginal). In [5] there is the method of a circular wire over a flat plane described. It consists of two steps. The first of them is a scaling (see below) and the second one is an inverse transform to (3) – regress back to the plane w

$$w^d = \operatorname{arctanh} p^d.$$ (4)

The whole magic is in a smart scaling. The requested result we will get if the point $p_1^d \to w_1^d \to \infty$ is the geometrical average value of $p_2 p_5$

$$p_1^b = \sqrt{p_2 p_5},$$ (5)

but the point $w_1^d \to \infty$ is the map (4) of the point $p_1^d = 1$. So if we adjust the plane p^b by division by p_1^b, we get the plane p^d we have wanted

$$p^d = p^b / p_1^b.$$ (6)

We can say we stretch the shape in the plane p^b by the factor $1/p_1^b$ to get the shape in the plane p^d.

The described method corrects the main distortion of the rectangular shape of our structure in the plane w. Because the arc $p_2 p_5$ is not exactly a half circle, the curve $w_2^d w_5^d$ is not a straight abscissa, but the deviation is neglectable (except the limit case when z_4 is very close to 1).

2.3 The Application of CC on SCCL

Let us have "an ideal transmission line" - the rectangle with two opposite electrodes made from PEC (their proportion is y) and with the rest sides (proportion x) made from PMC. The direction of the electrical field intensity vector is perpendicular to electrodes and the vector of magnetic field intensity is parallel to them (Fig. 3.). The field inside the rectangle is homogenous. The characteristic wave impedance is

$$Z_0 = \frac{E_x \cdot x}{H_y \cdot y} = \frac{120\pi}{\sqrt{\varepsilon_r}} \frac{x}{y}.$$ (7)

The shape on Fig. 2.d is approximately such a structure with $x = w_5^d$ and $y = \pi/2$, but because we use only one quarter of the whole cross section for the analysis and all four quarters are connected "in parallel" in full cross section of the SCCL, the true characteristic impedance value is 4 times smaller

$$Z_0 = \frac{1}{4} \frac{120\,\pi}{\sqrt{\varepsilon_r}} \frac{w_5^d}{0.5\,\pi} = \frac{60}{\sqrt{\varepsilon_r}} w_5^d \ . \qquad (8)$$
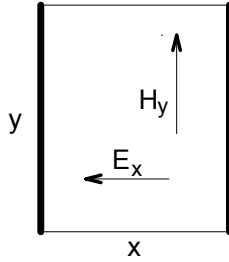

Fig. 3. Ideal transmission line (thick line: PEC, thin line: PMC).

The aim of the next paragraph is to find a formula for the description of w_5^d. From (2) - (6) we get

$$w_2^b = \operatorname{arccosh} \frac{z_2}{z_4} = \operatorname{arccosh} \frac{j}{z_4},$$

$$w_5^b = \operatorname{arccosh} \frac{z_5}{z_4} = \operatorname{arccosh} \frac{1}{z_4},$$

$$p_2^b = \tanh w_2^b, \qquad\qquad (9\text{a-f})$$

$$p_5^b = \tanh w_5^b,$$

$$p_5^d = \frac{p_5^b}{\sqrt{p_2^b p_5^b}} = \sqrt{\frac{p_5^b}{p_2^b}},$$

$$w_5^d = \operatorname{arctanh} p_5^d.$$

The formulae (9a) and (9b) after the substitution

$$\operatorname{arccosh} x = \operatorname{arctanh} \frac{\sqrt{x^2-1}}{x} \ ; \ \ x = \frac{1}{z_4}$$

and some derivations are

$$w_2^b = \operatorname{arctanh} \sqrt{1+z_4^2},$$
$$w_5^b = \operatorname{arctanh} \sqrt{1-z_4^2}. \qquad (10\text{a, b})$$

So in the plane $\boldsymbol{p}^b$ we get

$$p_2^b = \sqrt{1+z_4^2},$$
$$p_5^b = \sqrt{1-z_4^2}. \qquad (11\text{a, b})$$

After the CC (9e) the point p_5^b moves to

$$p_5^d = \sqrt{\frac{\sqrt{1-z_4^2}}{\sqrt{1+z_4^2}}} = \sqrt[4]{\frac{1-z_4^2}{1+z_4^2}}. \qquad (12)$$

Transform (4) gives us the location of the point w_5^d

$$w_5^d = \operatorname{arctanh} p_5^d = \operatorname{arctanh} \sqrt[4]{\frac{1-z_4^2}{1+z_4^2}}. \qquad (13)$$

and the installation of (13) into (8) faces to characteristic impedance formula of the SCCL

$$Z_0 = \frac{60}{\sqrt{\varepsilon_r}} \operatorname{arctanh} \sqrt[4]{\frac{1-z_4^2}{1+z_4^2}} \ . \qquad (14)$$

3. Numerical Solution

Because the SCCL is not a commonly used structure, it is very difficult to verify the results of the described analysis. That is the reason why two different numerical models are used.

3.1 Numerical Solution of Schwarz-Christoffel Integral

This method is based on the same theoretical basis as above presented solution, but solved numerically.

A practical tool for numerical Schwarz-Christoffel integral enumeration is the *sc-toolbox* [6] for *Matlab®*. Because of this toolbox works only with straight lines, the arc $z_2 z_5$ is approximated by a piecewise broken line.

List of main part of the *m-file*:

```
a=linspace(0, -i*pi/2,N)';
p=polygon([0; z4; exp(a)]);
f=rectmap(p,[1 2 3 length(p)]);
m=modulus(f);
Zsc=30*pi/m;
```

The symbol `z4` means the half width of the strip z_4 according to Fig. 2 and `N` is the number of broken line pieces. In the enumeration the value $N = 20$ was selected.

3.2 Boundary Element Method

The boundary element method (BEM) [7] is based on integral Maxwell's equations (Laplace's equation is solved along the boundary of the structure only). We need values of the field intensity just along PEC's for characteristic impedance determination according to the formula below:

$$Z_0 = \frac{\int E\,dy}{\int H\,dx} = 30\,\pi\,U \Big/ \int_0^{z_4} E_y\,dx \qquad (15)$$

where U is the voltage between conductors and E_y is the electric field y-component along the central electrode. (The integrals in the first fraction are only informative; they both are not displayed in the exact form)

The boundary $z_2 z_3 z_4 z_5$ (Fig. 2a) is divided into boundary elements, with Dirichlet's condition along PEC and Neumann's condition along PMC. The vectors of potentials in all boundary elements and the vectors of their gradients (both vectors partly known and partly not known) are mutually tied by two matrices H and G, built up according the rules of BEM.

The task was solved in *Matlab®* again. Two different strategies were tested.

The first of them is based on a constant number of boundary elements ($n = 50$) per one basic element. The flat inner electrode is divided into $n = 50$ elements, the round outer electrode is divided also into $n = 50$ elements and both the magnetic walls are also divided each into $n = 50$ pieces. That means the total sum of all boundary elements is $4n = 200$. The lengths of elements are not the same, especially in case of $z_4 \neq 0.5$.

The second strategy is based on equidistant division of the border (except the outer electrode). The inner electrode is divided to z_4 times n pieces, the continuing magnetic wall into $(1 - z_4)$ times n pieces, the outer electrode into n elements and the last magnetic wall also into n elements. On the ground of calculation in a wide range of z_4, the number of elements was $n = 100$.

Z_0 [Ω] / error [%]					
z_4	This work	Wadell [3]	sc–toolbox	BEM-1	BEM-2
0.01	317.89 0	317.89	317.86 0.01	651.46 105	323.41 1.73
0.05	221.33 0	221.33	221.31 0.02	273.26 23.46	223.21 0.85
0.1	179.74 0	179.74	179.71 0.02	206.83 15.07	180.75 0.55
0.2	138.14 0.01	138.16	138.11 0.03	152.71 10.53	138.64 0.35
0.3	113.75 0.07	113.83	113.73 0.08	122.83 7.52	114.10 0.25
0.4	96.32 0.25	96.57	96.34 0.24	96.53 0.03	96.63 0.06
0.5	82.58 0.06	82.63	82.66 0.04	82.89 0.32	82.91 0.34
0.6	70.95 0.34	71.19	71.17 0.03	71.42 0.32	71.39 0.28
0.7	60.47 0.83	60.98	60.94 0.06	60.98 0.38	61.15 0.28
0.8	50.25 1.89	51.22	51.17 0.08	51.48 0.50	51.38 0.32
0.9	38.78 4.71	40.70	40.64 0.15	41.03 0.81	40.89 0.48
0.95	31.05 8.62	33.98	33.89 0.25	34.44 1.35	34.24 0.76
0.99	19.67 20.45	24.73	24.57 0.64	26.14 5.71	25.39 2.68

Tab. 1. Results of analysis (error compared to Wadell [3]).

4. Results

Results of all methods are displayed in Tab. 1.

Since this work focuses on SCCL analysis without a special analysis of the dielectric substrate (homogeneity assumption), results were calculated for free space ($\varepsilon_r = 1$).

All results are compared to the relevant results of the method mentioned in Wadell's book [3], however he does not inform about the accuracy of the mentioned formula. He only cites his sources – Bongianni [8] and Hilberg [9]. Bongianni made an experiment which confirms this method (the accuracy is in range of measurement errors), Hilberg [9] is the author of this method, but he only describes it, not comparing to any other method. But only these results [3] (and also the same in [8] and [9]) are independent on the author of this work.

5. Conclusion

One method founded in literature [3] is easy, but with the chicken-to-egg problem of choosing one of two formulae according to their results.

The designed method has no chicken-to-egg problem, but it is little bit complicated in mathematical way - it contains the biquadratic root and arc-hyperbolic function.

The error of the designed method compared to [3] is very small, grows only for z_4 very close to 1. The reason of this error is in curveting of the wall $w_2 w_5$ which grows for $z_4 \rightarrow 1$ (see Fig. 2.d). In consequence of nonzero thickness of the inner conductor the reliability of all methods in this region is pure. Moreover the characteristic impedance of the real structure is in this case smaller than the result of [3] (nonzero thickness of the central strip) and the described method results are also smaller. They are maybe mathematically incorrect, but more in consonance of physical reality.

Acknowledgement

References

[1] SVAČINA, J. *Electromagnetic Compatibility – lectures.* Textbook of the Brno University of Technology, FEKT VUT Brno, 2002.

[2] ARMSTRONG, K. *Design Techniques for EMC.* http://www.compliance-club.com/keith_armstrong.asp

[3] WADELL, B. C. *Transmission Line Design Handbook.* Boston-London: Artech House, 1991.

[4] SVAČINA, J. *Investigation of Hybrid Microwave Integrated Circuits by a Conformal Mapping Method).* Brno Univ. of Technology, A-46, Brno, 1991 (in Czech).

[5] ŠÁDEK, V. Analysis of the two-wire transmission line by a conformal mapping method. In *Proc. of Radioelektronika*, Bratislava 2002.

[6] DRISCOLL, T. A. *SC-Tolbox, for MATLAB.* http://www.math.edu/~driscoll/SC/

[7] PARIS, F., CANAS, J. *Boundary Element Method.* Oxford: Oxford University Press, 1997.

[8] BONGIANNI, W. L. Fabrication and performance of strip-centered microminiature coaxial cable. In *Proceeding of the IEEE,* 1984, vol. 72, no. 12, p. 1810-1811.

[9] HILBERG, W. *Electric Characteristics of Transmission Lines.* Dedham, MA: Artech House, 1979, p. 122.