Computer Simulation of Nonuniform MTLs via Implicit Wendroff and State-Variable Methods

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Abstract. *The paper deals with techniques for a computer simulation of nonuniform multiconductor transmission lines (MTLs) based on the implicit Wendroff and the state-variable methods. The techniques fall into a class of finite-difference time-domain (FDTD) methods useful to solve various electromagnetic systems. Their basic variants are extended and modified to enable solving both voltage and current distributions along nonuniform MTL's wires and their sensitivities with respect to lumped and distributed parameters. An experimental error analysis is performed based on the Thomson cable whose analytical solutions are known, and some examples of simulation of both uniform and nonuniform MTLs are presented. Based on the Matlab language programme, CPU times are analyzed to compare efficiency of the methods. Some results for nonlinear MTLs simulation are presented as well.*

Keywords

Multiconductor transmission line, Wendroff method, state-variable method, sensitivity analysis, Matlab.

1. Introduction

The simulation of multiconductor transmission lines (MTL) plays an important role in a design of today's high-speed electronic systems, especially due to continuously increasing clock frequencies resulting in signal integrity problems at the transmission structures [1]. Here not only possibilities to evaluate waveforms of voltage or current signals propagated, but also to determine their sensitivities with respect to various parameters are needed to be able to optimize the design. Besides more frequent uniform MTLs, the nonuniform ones should sometimes be considered to model more general transmission structures.

The paper focuses its attention to two principles of the computer simulation of the nonuniform MTLs. First, the implicit Wendroff method [2], [3], originally used to solve transient phenomena on single and three-phase TLs in field of a power engineering, is discussed and further extended to enable such the solution. The uniform MTLs considered in [4] have served as a starting point for generalizing the method towards nonuniform MTLs [5]. To show further potential of the method, first experiments with a simulation of nonlinear MTLs are also shortly discussed. Second, the state-variable methods [6], [7], [8] are presented and extended for nonuniform MTLs simulation, here both in the time and the Laplace domain. The latter is connected with a proper technique of the numerical inversion of Laplace transforms to get the required time-domain solution. To evaluate computational efficiencies of the methods the CPU times have been assessed depending on the number of MTL's wires and points of discretization. Both approaches are in relation to a broad class of similar finite-difference time-domain (FDTD) techniques being elaborated for solving various electromagnetic systems in the time domain, [9], [10], including the MTLs, [11], [12].

Let us consider a simple MTL system containing an $(n+1)$-conductor transmission line, terminated by lumped-parameter circuits, left (L), right (R), as shown in Fig. 1.

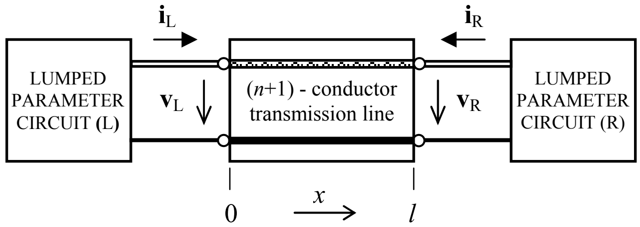

Fig. 1. MTL system containing $(n+1)$-conductor transmission line.

Let us first consider a linear MTL defined by its length l and per-unit-length (p.-u.-l.) $n \times n$ matrices $\mathbf{R}_0(x)$, $\mathbf{L}_0(x)$, $\mathbf{G}_0(x)$ and $\mathbf{C}_0(x)$, i.e. nonuniform in general. The MTL telegraphic equations are [13]

$$\begin{aligned} -\frac{\partial \mathbf{v}(t,x)}{\partial x} &= \mathbf{R}_0(x)\mathbf{i}(t,x) + \mathbf{L}_0(x)\frac{\partial \mathbf{i}(t,x)}{\partial t} \ , \\ -\frac{\partial \mathbf{i}(t,x)}{\partial x} &= \mathbf{G}_0(x)\mathbf{v}(t,x) + \mathbf{C}_0(x)\frac{\partial \mathbf{v}(t,x)}{\partial t} \ , \end{aligned} \tag{1}$$

where $\mathbf{v}(t,x)$ and $\mathbf{i}(t,x)$ are $n \times 1$ column vectors of voltages and currents of n active wires at the distance x from the MTL's left end, respectively. Equation (1) is supplemented by boundary conditions reflecting terminating lumped-parameter circuits, by using their generalized Thévenin or Norton equivalents, for example, as will be shown later.

2. MTL via Implicit Wendroff Method

The principle of the implicit Wendroff formula lies in following operations on (1). For the j-th time step and k-th spatial coordinate, (1) is modified by substitutions

$$\left.\frac{\partial \mathbf{u}(t,x)}{\partial t}\right|_{j,k} \doteq \frac{1}{2}\left(\frac{\mathbf{u}_k^j-\mathbf{u}_k^{j-1}}{\Delta t}+\frac{\mathbf{u}_{k+1}^j-\mathbf{u}_{k+1}^{j-1}}{\Delta t}\right),$$
$$\left.\frac{\partial \mathbf{u}(t,x)}{\partial x}\right|_{j,k} \doteq \frac{1}{2}\left(\frac{\mathbf{u}_{k+1}^j-\mathbf{u}_k^{j}}{\Delta x}+\frac{\mathbf{u}_{k+1}^{j-1}-\mathbf{u}_{k}^{j-1}}{\Delta x}\right), \qquad (2)$$
$$\left.\mathbf{u}(t,x)\right|_{j,k} \doteq \frac{1}{4}\left(\mathbf{u}_{k+1}^j+\mathbf{u}_k^j+\mathbf{u}_{k+1}^{j-1}+\mathbf{u}_k^{j-1}\right),$$

where $\mathbf{u}(t,x)$ denotes either the voltage $\mathbf{v}(t,x)$ or current $\mathbf{i}(t,x)$ vectors. The indexes have ranges $k = 1,2,...,K$, and $j = 1,2,...,J$, with K and J as the numbers of intervals $\Delta x = l/K$ and $\Delta t = T/J$ in space and time, respectively, and where l denotes the MTL's length and T the upper limit of the time interval of interest. So we have chosen equidistant intervals to simplify the notation, although they could vary in general. Substituting (2) into (1) leads to

$$\mathbf{v}_k^j-\mathbf{v}_{k+1}^j+\mathbf{A}_{\mathrm{v}k}(\mathbf{i}_k^j+\mathbf{i}_{k+1}^j)=-\mathbf{v}_k^{j-1}+\mathbf{v}_{k+1}^{j-1}+\mathbf{B}_{\mathrm{v}k}(\mathbf{i}_k^{j-1}+\mathbf{i}_{k+1}^{j-1}),$$
$$\mathbf{A}_{\mathrm{i}k}(\mathbf{v}_k^j+\mathbf{v}_{k+1}^j)+\mathbf{i}_k^j-\mathbf{i}_{k+1}^j=\mathbf{B}_{\mathrm{i}k}(\mathbf{v}_k^{j-1}+\mathbf{v}_{k+1}^{j-1})-\mathbf{i}_k^{j-1}+\mathbf{i}_{k+1}^{j-1}, \qquad (3)$$

with coefficients matrices

$$\mathbf{A}_{\mathrm{v}k}=-\left(\frac{\mathbf{R}_{0k}}{2}+\frac{\mathbf{L}_{0k}}{\Delta t}\right)\Delta x\;,\;\mathbf{B}_{\mathrm{v}k}=\left(\frac{\mathbf{R}_{0k}}{2}-\frac{\mathbf{L}_{0k}}{\Delta t}\right)\Delta x\,,$$
$$\mathbf{A}_{\mathrm{i}k}=-\left(\frac{\mathbf{G}_{0k}}{2}+\frac{\mathbf{C}_{0k}}{\Delta t}\right)\Delta x\;,\;\mathbf{B}_{\mathrm{i}k}=\left(\frac{\mathbf{G}_{0k}}{2}-\frac{\mathbf{C}_{0k}}{\Delta t}\right)\Delta x\,. \qquad (4)$$

Here $\mathbf{R}_{0k} = \mathbf{R}_0(\xi_k)$, $\mathbf{L}_{0k} = \mathbf{L}_0(\xi_k)$, $\mathbf{G}_{0k} = \mathbf{G}_0(\xi_k)$, $\mathbf{C}_{0k} = \mathbf{C}_0(\xi_k)$, with $\xi_k \in (x_k, x_{k+1})$, often $\xi_k = (x_k + x_{k+1})/2$. Defining further

$$\mathbf{v}^j=\left[\mathbf{v}_1^{j\mathrm{T}},\mathbf{v}_2^{j\mathrm{T}},\ldots,\mathbf{v}_{K+1}^{j\mathrm{T}}\right]^{\mathrm{T}}\;,\;\mathbf{i}^j=\left[\mathbf{i}_1^{j\mathrm{T}},\mathbf{i}_2^{j\mathrm{T}},\ldots,\mathbf{i}_{K+1}^{j\mathrm{T}}\right]^{\mathrm{T}}, \qquad (5)$$

as column vectors of the order $n(K+1) \times 1$, and finally

$$\mathbf{x}^j=\left[\mathbf{v}^{j\mathrm{T}},\mathbf{i}^{j\mathrm{T}}\right]^{\mathrm{T}} \qquad (6)$$

as the $2n(K+1) \times 1$ column vector, T as the transposition, we can write a recursive formula

$$\mathbf{x}^j=\mathbf{A}^{-1}\left(\mathbf{B}\mathbf{x}^{j-1}+\mathbf{D}^j\right). \qquad (7)$$

Formula (7) expresses the solution in incoming time t_j, based on the values in preceding time t_{j-1}. The matrices **A** and **B** are formed by (4) and by boundary conditions, the column vector $\mathbf{D}^j$ depends on values of external sources taken in time t_j. A constitution of the matrices is explained by (8), when the MTL is divided only on $K = 3$ parts.

The terminating circuits are supposed linear resistive, and they can be replaced by their generalized Thévenin equivalents with matrices of internal resistances $\mathbf{R}_{\mathrm{iL}}$ and $\mathbf{R}_{\mathrm{iR}}$, and vectors of internal voltages $\mathbf{v}_{\mathrm{iL}}(t)$ and $\mathbf{v}_{\mathrm{iR}}(t)$.

$$\begin{bmatrix}\mathbf{v}_1\\\mathbf{v}_2\\\mathbf{v}_3\\\mathbf{v}_4\\\mathbf{i}_1\\\mathbf{i}_2\\\mathbf{i}_3\\\mathbf{i}_4\end{bmatrix}^j=\left[\begin{array}{cccc|cccc}\mathbf{I}&\mathbf{-I}&\mathbf{0}&\mathbf{0}&\mathbf{A}_{\mathrm{v1}}&\mathbf{A}_{\mathrm{v1}}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{I}&\mathbf{-I}&\mathbf{0}&\mathbf{0}&\mathbf{A}_{\mathrm{v2}}&\mathbf{A}_{\mathrm{v2}}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{I}&\mathbf{-I}&\mathbf{0}&\mathbf{0}&\mathbf{A}_{\mathrm{v3}}&\mathbf{A}_{\mathrm{v3}}\\\hline\mathbf{A}_{\mathrm{i1}}&\mathbf{A}_{\mathrm{i1}}&\mathbf{0}&\mathbf{0}&\mathbf{I}&\mathbf{-I}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{A}_{\mathrm{i2}}&\mathbf{A}_{\mathrm{i2}}&\mathbf{0}&\mathbf{0}&\mathbf{I}&\mathbf{-I}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{A}_{\mathrm{i3}}&\mathbf{A}_{\mathrm{i3}}&\mathbf{0}&\mathbf{0}&\mathbf{I}&\mathbf{-I}\\\hline\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{R}_{\mathrm{iL}}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{-R}_{\mathrm{iR}}\end{array}\right]^{-1}\times$$
$$\times\left(\left[\begin{array}{cccc|cccc}\mathbf{-I}&\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{B}_{\mathrm{v1}}&\mathbf{B}_{\mathrm{v1}}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{-I}&\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{B}_{\mathrm{v2}}&\mathbf{B}_{\mathrm{v2}}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{-I}&\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{B}_{\mathrm{v3}}&\mathbf{B}_{\mathrm{v3}}\\\hline\mathbf{B}_{\mathrm{i1}}&\mathbf{B}_{\mathrm{i1}}&\mathbf{0}&\mathbf{0}&\mathbf{-I}&\mathbf{I}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{B}_{\mathrm{i2}}&\mathbf{B}_{\mathrm{i2}}&\mathbf{0}&\mathbf{0}&\mathbf{-I}&\mathbf{I}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{B}_{\mathrm{i3}}&\mathbf{B}_{\mathrm{i3}}&\mathbf{0}&\mathbf{0}&\mathbf{-I}&\mathbf{I}\\\hline\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{array}\right]\cdot\begin{bmatrix}\mathbf{v}_1\\\mathbf{v}_2\\\mathbf{v}_3\\\mathbf{v}_4\\\mathbf{i}_1\\\mathbf{i}_2\\\mathbf{i}_3\\\mathbf{i}_4\end{bmatrix}^{j-1}+\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{v}_{\mathrm{iL}}\\\mathbf{v}_{\mathrm{iR}}\end{bmatrix}^j\right) \qquad (8)$$

They are located in two downmost lines of (8) accordant with the boundary conditions incorporation. Generally, when reactive elements are included, ordinary differential equations will be considered instead of the algebraic ones. To test computational efficiencies of the methods in view, just resistive circuits will be used. Finally, **I** and **0** are the n-th order identity and zero matrices, respectively.

Equation (7) will now be used to derive a formula for evaluation of sensitivities. Let us consider a parameter γ which can be an element of the MTL p.-u.-l. matrices, the MTL length l or some lumped parameter of the terminating circuits. After some arrangements, we have

$$\frac{\partial \mathbf{x}^j}{\partial\gamma}=\mathbf{A}^{-1}\left(\mathbf{B}\frac{\partial \mathbf{x}^{j-1}}{\partial\gamma}-\frac{\partial \mathbf{A}}{\partial\gamma}\mathbf{x}^j+\frac{\partial \mathbf{B}}{\partial\gamma}\mathbf{x}^{j-1}+\frac{\partial \mathbf{D}^j}{\partial\gamma}\right). \qquad (9)$$

It is clear (7) is also needed in (9) thus their simultaneous recursive processing leads to both voltage and current distributions and their sensitivities. Firstly, if we exclude sensitivities with respect to voltages of external sources, $\partial\mathbf{D}^j/\partial\gamma = \mathbf{0}$. The $\partial\mathbf{A}/\partial\gamma$ and $\partial\mathbf{B}/\partial\gamma$ are computed submatrix-wise as is obvious from their decomposed forms in (8). Finally, $\partial\mathbf{R}_{\mathrm{iL(R)}}/\partial\gamma = \mathbf{0}$ if γ is a distributed parameter, see Tab. 1, or it is determined easily if γ is a lumped parameter.

Parameter	$\frac{\partial \mathbf{A}_{\mathrm{v}k}}{\partial\gamma}$	$\frac{\partial \mathbf{B}_{\mathrm{v}k}}{\partial\gamma}$	$\frac{\partial \mathbf{A}_{\mathrm{i}k}}{\partial\gamma}$	$\frac{\partial \mathbf{B}_{\mathrm{i}k}}{\partial\gamma}$
$\gamma\in\mathbf{C}_{0k}$	$\mathbf{0}$	$\mathbf{0}$	$-\frac{\partial \mathbf{C}_{0k}}{\partial\gamma}\frac{\Delta x}{\Delta t}$	$-\frac{\partial \mathbf{C}_{0k}}{\partial\gamma}\frac{\Delta x}{\Delta t}$
$\gamma\in\mathbf{L}_{0k}$	$-\frac{\partial \mathbf{L}_{0k}}{\partial\gamma}\frac{\Delta x}{\Delta t}$	$-\frac{\partial \mathbf{L}_{0k}}{\partial\gamma}\frac{\Delta x}{\Delta t}$	$\mathbf{0}$	$\mathbf{0}$
$\gamma\in\mathbf{G}_{0k}$	$\mathbf{0}$	$\mathbf{0}$	$-\frac{\partial \mathbf{G}_{0k}}{\partial\gamma}\frac{\Delta x}{2}$	$\frac{\partial \mathbf{G}_{0k}}{\partial\gamma}\frac{\Delta x}{2}$
$\gamma\in\mathbf{R}_{0k}$	$-\frac{\partial \mathbf{R}_{0k}}{\partial\gamma}\frac{\Delta x}{2}$	$\frac{\partial \mathbf{R}_{0k}}{\partial\gamma}\frac{\Delta x}{2}$	$\mathbf{0}$	$\mathbf{0}$
$\gamma\equiv l$	$\frac{\mathbf{A}_{\mathrm{v}k}}{l}$	$\frac{\mathbf{B}_{\mathrm{v}k}}{l}$	$\frac{\mathbf{A}_{\mathrm{i}k}}{l}$	$\frac{\mathbf{B}_{\mathrm{i}k}}{l}$

Tab. 1. Derivatives of matrices (4) with respect to parameters γ.

3. MTL via State-Variable Method

While the implicit Wendroff method represents a fully discrete MTL model, when both geometrical and time variables are discretized simultaneously, a model based on a state variable method ranks among semidiscrete ones as only geometrical variable is discretized at the first step. A corresponding MTL circuit model, a cascade connection of generalized Π networks, is shown in Fig. 2, with possible external circuits connected.

The voltage vectors $\mathbf{v}_k$, $k = 1,...,K+1$, with $\mathbf{v}_1 \equiv \mathbf{v}_L$ and $\mathbf{v}_{K+1} \equiv \mathbf{v}_R$, and the current vectors $\mathbf{i}_k$, $k = 1,...,K$, are the vectors of state variables unknown. The lumped parameters of the model are set up by $\mathbf{C}_{dk} = \mathbf{C}_0(x_k)\Delta x$, $\mathbf{G}_{dk} = \mathbf{G}_0(x_k)\Delta x$, $\mathbf{L}_{dk} = \mathbf{L}_0(x_k+\Delta x/2)\Delta x$, $\mathbf{R}_{dk} = \mathbf{R}_0(x_k+\Delta x/2)\Delta x$, with $\Delta x = l/K$, and l as the MTL length. The current vectors of external sources are stated by $\mathbf{i}_{ik} = \mathbf{R}_{ik}^{-1}(\mathbf{v}_{ik} - \mathbf{v}_k)$, when the internal Thévenin matrices are supposed as regular, i.e. $\mathbf{G}_{ik} = \mathbf{R}_{ik}^{-1}$ exists. To explain a constitution of state equations, $K = 2$ is considered, while mixed cut-set and loop analyses lead to

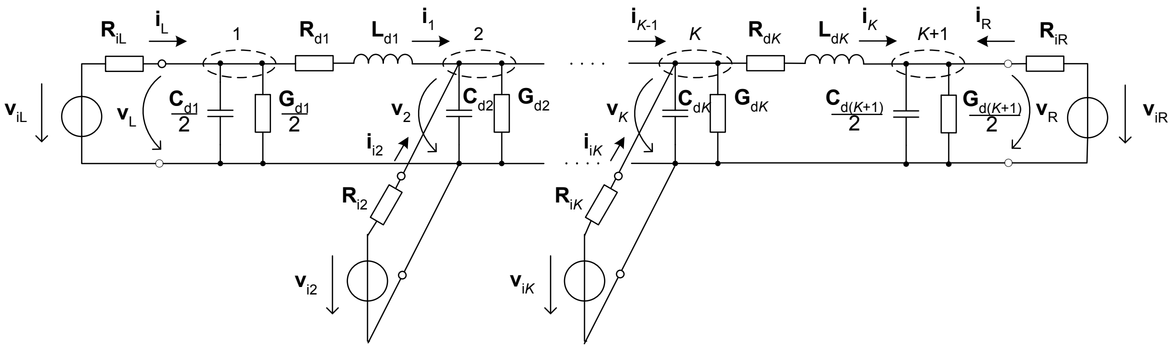

Fig. 2. MTL semidiscrete model for state-variable method based on generalized Π sections.

$$\left[\begin{array}{ccc|cc} \mathbf{C}_{d1}/2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{d2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{d3}/2 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}_{d1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}_{d2} \end{array}\right] \cdot \frac{d}{dt} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{i}_1 \\ \mathbf{i}_2 \end{bmatrix} = \\ -\left(\left[\begin{array}{ccc|cc} \mathbf{G}_{d1}/2 & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{d2} & \mathbf{0} & -\mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{d3}/2 & \mathbf{0} & -\mathbf{I} \\ \hline -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{R}_{d1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{R}_{d2} \end{array}\right] + \right. \\ \left. + \left[\begin{array}{ccc|cc} \mathbf{G}_{i1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{i2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{i3} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \right) \cdot \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{i}_1 \\ \mathbf{i}_2 \end{bmatrix} + \left[\begin{array}{ccc|cc} \mathbf{G}_{i1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{i2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}_{i3} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \cdot \begin{bmatrix} \mathbf{v}_{i1} \\ \mathbf{v}_{i2} \\ \mathbf{v}_{i3} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (10)$$

Based on (10), a general description can be written as

$$\mathbf{M}\frac{d\mathbf{x}(t)}{dt} = -\left(\mathbf{H}+\mathbf{P}\right)\mathbf{x}(t) + \mathbf{P}\mathbf{u}(t) \quad (11)$$

with

$$\mathbf{x}(t) = \left[\mathbf{v}_C^T(t), \mathbf{i}_L^T(t)\right]^T \quad (12)$$

as the vector of unknown state variables. Generally, for the K-sectional model, we have $n(2K+1)$ elements inside $\mathbf{x}(t)$, grouped into $n \times 1$ column vectors, namely $\mathbf{v}_C(t)$ holds $K+1$ vectors of state voltages and $\mathbf{i}_L(t)$ holds K vectors of state currents. The memory $\mathbf{M}$ and memoryless $\mathbf{H}$ matrices are formed based on (10) via the MTL p.-u.-l. matrices

$$\mathbf{M} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{G} & \mathbf{E} \\ \mathbf{-E}^T & \mathbf{R} \end{bmatrix}, \quad (13)$$

with block-diagonal matrices

$$\mathbf{C} = \operatorname{diag}\left(\mathbf{C}_{d1}/2, \mathbf{C}_{d2}, \ldots, \mathbf{C}_{dK}, \mathbf{C}_{d(K+1)}/2\right) \\ \mathbf{L} = \operatorname{diag}\left(\mathbf{L}_{d1}, \ldots, \mathbf{L}_{dK}\right) \quad (14a)$$

$$\mathbf{G} = \operatorname{diag}\left(\mathbf{G}_{d1}/2, \mathbf{G}_{d2}, \ldots, \mathbf{G}_{dK}, \mathbf{G}_{d(K+1)}/2\right) \\ \mathbf{R} = \operatorname{diag}\left(\mathbf{R}_{d1}, \ldots, \mathbf{R}_{dK}\right) \quad (14b)$$

and $\mathbf{E}$ as the diagonal-constant matrix, namely the block-bidiagonal matrix formed by n-th order identity matrices according to (10). The coefficients matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{Y}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (15)$$

is dependent on the external terminating circuits, namely $\mathbf{Z}_i = \operatorname{diag}(\mathbf{R}_{i1},...,\mathbf{R}_{i(K+1)})$ is a block-diagonal matrix formed by Thévenin resistive matrices, and the column vector

$$\mathbf{u}(t) = \left[\mathbf{v}_i^T(t), \mathbf{0}\right]^T \quad (16)$$

contains internal voltages $\mathbf{v}_i(t) = [\mathbf{v}_{i1}(t),..., \mathbf{v}_{i(K+1)}(t)]^T$.

3.1 Direct Time-Domain Solution

The first-order ordinary matrix differential equation (11) has the well-known solution in the time-domain [14]

$$\mathbf{x}(t) = e^{-\mathbf{M}^{-1}(\mathbf{H}+\mathbf{P})t}\mathbf{x}(0) + \int_0^t e^{-\mathbf{M}^{-1}(\mathbf{H}+\mathbf{P})(t-\tau)}\mathbf{M}^{-1}\mathbf{P}\mathbf{u}(\tau)d\tau \quad (17)$$

when choosing $t = 0$ as the initial instant of time. To evaluate this formula two key parts must be solved: the matrix exponential function and the convolution integral. Both are becoming harder to perform due to large orders of matrices used. From computational point of view it is therefore more advantageous to use a procedure as follows. When choosing an equidistant time division $\Delta t = t_j - t_{j-1}$, $\forall j$, and using the rectangular rule of the integration, an approximate recursive formula can be developed as

$$\mathbf{x}^j = e^{\mathbf{A}\Delta t}\mathbf{x}^{j-1} + (\mathbf{I} - e^{\mathbf{A}\Delta t})\mathbf{B}\mathbf{u}^j \tag{18}$$

where denotations were introduced as

$$\mathbf{A} = -\mathbf{M}^{-1}\left(\mathbf{H}+\mathbf{P}\right)\ , \quad \mathbf{B} = \left(\mathbf{H}+\mathbf{P}\right)^{-1}\mathbf{P}\ , \tag{19}$$

with $\mathbf{x}^j \approx \mathbf{x}(t_j)$, $\mathbf{u}^j = \mathbf{u}(t_j)$ and $\mathbf{I}$ denoting the identity matrix. In case of $\mathbf{u}(t) = 0$, i.e. when finding response to the initial condition $\mathbf{x}(0) \neq 0$ only, the recursive formula leads to the exact solution $\mathbf{x}^j = \mathbf{x}(t_j)$, independently on a choice of Δt. The matrix exponential function can be stated by various techniques, e.g. via the Taylor series expansion [15].

To find sensitivity with respect to parameter γ we can proceed as follows. First, if we exclude sensitivities with respect to the elements of external circuits voltages $\mathbf{u}^j$, i.e. when $\partial\mathbf{u}^j/\partial\gamma = \mathbf{0}$, (18) can be differentiated leading to

$$\frac{\partial\mathbf{x}^j}{\partial\gamma} = e^{\mathbf{A}\Delta t}\frac{\partial\mathbf{x}^{j-1}}{\partial\gamma} + \frac{\partial e^{\mathbf{A}\Delta t}}{\partial\gamma}\mathbf{x}^{j-1} + \left((\mathbf{I} - e^{\mathbf{A}\Delta t})\frac{\partial\mathbf{B}}{\partial\gamma} - \frac{\partial e^{\mathbf{A}\Delta t}}{\partial\gamma}\mathbf{B}\right)\mathbf{u}^j\ . \tag{20}$$

It is clear that (20) must be evaluated together with (18). To get a derivative of a matrix exponential function in (20), more techniques can be used [16]. For each one, however, the derivative of $\mathbf{A}$ in the exponent is needed. Hence, both $\partial\mathbf{A}/\partial\gamma$ and $\partial\mathbf{B}/\partial\gamma$ are shown in Tab. 2 depending on γ.

Parameter	$\frac{\partial\mathbf{A}}{\partial\gamma}$	$\frac{\partial\mathbf{B}}{\partial\gamma}$
$\gamma \in \mathbf{M}$	$-\mathbf{M}^{-1}\frac{\partial\mathbf{M}}{\partial\gamma}\mathbf{A}$	$\mathbf{0}$
$\gamma \in \mathbf{H}$	$-\mathbf{M}^{-1}\frac{\partial\mathbf{H}}{\partial\gamma}$	$-\left(\mathbf{H}+\mathbf{P}\right)^{-1}\frac{\partial\mathbf{H}}{\partial\gamma}\mathbf{B}$
$\gamma \equiv l$	$-\mathbf{M}^{-1}\left(\frac{\partial\mathbf{M}}{\partial l}\mathbf{A} + \frac{\partial\mathbf{H}}{\partial l}\right)$	$-\left(\mathbf{H}+\mathbf{P}\right)^{-1}\frac{\partial\mathbf{H}}{\partial l}\mathbf{B}$
$\gamma \in \mathbf{P}$	$-\mathbf{M}^{-1}\frac{\partial\mathbf{P}}{\partial\gamma}$	$\left(\mathbf{H}+\mathbf{P}\right)^{-1}\frac{\partial\mathbf{P}}{\partial\gamma}(\mathbf{I}-\mathbf{B})$

Tab. 2. Derivatives of matrices (19) with respect to parameters γ.

Finally, the $\partial\mathbf{M}/\partial\gamma$ and $\partial\mathbf{H}/\partial\gamma$ are done submatrix-wise as follows from (13) and (14), see Tab. 3,

Parameter	$\frac{\partial\mathbf{M}}{\partial\gamma}$	$\frac{\partial\mathbf{H}}{\partial\gamma}$
$\gamma \in \mathbf{C}_0(x)$	$\begin{bmatrix}\partial\mathbf{C}/\partial\gamma & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$	$\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$
$\gamma \in \mathbf{L}_0(x)$	$\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \partial\mathbf{L}/\partial\gamma\end{bmatrix}$	$\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$
$\gamma \in \mathbf{G}_0(x)$	$\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$	$\begin{bmatrix}\partial\mathbf{G}/\partial\gamma & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$
$\gamma \in \mathbf{R}_0(x)$	$\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$	$\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \partial\mathbf{R}/\partial\gamma\end{bmatrix}$
$\gamma \equiv l$	$\begin{bmatrix}\mathbf{C}/l & \mathbf{0}\\ \mathbf{0} & \mathbf{L}/l\end{bmatrix}$	$\begin{bmatrix}\mathbf{G}/l & \mathbf{0}\\ \mathbf{0} & \mathbf{R}/l\end{bmatrix}$

Tab. 3. Derivatives of matrices (13) with respect to parameters γ.

and the $\partial\mathbf{P}/\partial\gamma$ follows (15) and results in

$$\frac{\partial\mathbf{P}}{\partial\gamma} = \begin{bmatrix}\frac{\partial\mathbf{Y}_\mathrm{i}}{\partial\gamma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} = \begin{bmatrix}-\mathbf{Z}_\mathrm{i}^{-1}\frac{\partial\mathbf{Z}_\mathrm{i}}{\partial\gamma}\mathbf{Z}_\mathrm{i}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{21}$$

with $\partial\mathbf{Z}_\mathrm{i}/\partial\gamma = \mathrm{diag}(\partial\mathbf{R}_{\mathrm{i}1}/\partial\gamma,\ldots,\partial\mathbf{R}_{\mathrm{i}(K+1)}/\partial\gamma)$ being the block-diagonal matrix.

3.2 Solution via Laplace Domain and NILT

Applying Laplace transformation onto (11) and doing some arrangements we get an s-domain solution

$$\mathbf{x}(s) = \left(\mathbf{H}+\mathbf{P}+s\mathbf{M}\right)^{-1}\left(\mathbf{M}\mathbf{x}_0 + \mathbf{P}\mathbf{u}(s)\right) \tag{22}$$

where $\mathbf{x}(s) = Ł\{\mathbf{x}(t)\}$ and $\mathbf{u}(s) = Ł\{\mathbf{u}(t)\}$ denote Laplace transforms of time-dependent variables, and $\mathbf{x}_0 = \mathbf{x}(t)|_{t=0}$ is the vector of initial conditions defined by (12). Note that the s-domain solution can be generalized towards the MTLs driven or terminated by memory-element circuits easily, through the matrix $\mathbf{P} \equiv \mathbf{P}(s)$, and possible frequency dependences of the MTL primary parameters could be incorporated, resulting in $\mathbf{M} \equiv \mathbf{M}(s)$ and $\mathbf{H} \equiv \mathbf{H}(s)$.

Herein formulae for the Laplace-domain sensitivities with respect to either MTL parameters or external circuits parameters can be stated. Namely, the differentiation of (22) with respect to a parameter γ leads to a formula

$$\begin{aligned}\frac{\partial\mathbf{x}(s)}{\partial\gamma} &= -\left(\mathbf{H}+\mathbf{P}+s\mathbf{M}\right)^{-1}\times\\ &\times\left(\frac{\partial\mathbf{H}}{\partial\gamma}\mathbf{x}(s) + \frac{\partial\mathbf{M}}{\partial\gamma}\left(s\mathbf{x}(s)-\mathbf{x}_0\right) + \frac{\partial\mathbf{P}}{\partial\gamma}\left(\mathbf{x}(s)-\mathbf{u}(s)\right)\right)\end{aligned} \tag{23}$$

The above derivatives are again stated via Tab. 3 and (21), in case of distributed- and lumped-parameter sensitivities, respectively. If zero initial conditions are considered, i.e. $\mathbf{x}_0 = \mathbf{0}$, (23) can slightly be simplified. The further solution continues with a usage of the numerical inversion of Laplace transforms (NILT) method to get the solution in the time domain, here [17] will be used in the examples.

4. Basic Experimental Error Analysis

An accuracy of the methods will experimentally be evaluated through known analytical solutions for voltage and current distributions on a Thomson cable, a uniform single TL with negligible L_0 and G_0 parameters. The cable is driven from a unit step voltage as shown in Fig. 3.

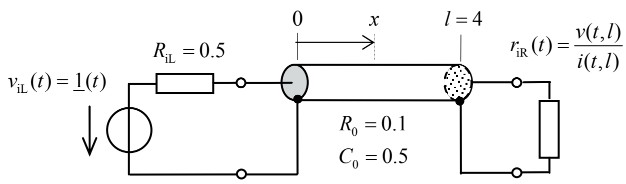

Fig. 3. Transmission line system with Thomson cable.

For an infinitely long cable, when no reflected waves exist, analytical solutions have the forms [7]

$$\begin{aligned} &i(t,x) = \underline{1}(t)/R_{\mathrm{iL}} \cdot e^{a(t)(a(t)+2b(t,x))} \cdot \operatorname{erfc}\left(a(t)+b(t,x)\right) \ , \\ &v(t,x) = \underline{1}(t) \cdot \operatorname{erfc}\left(b(t,x)\right) - R_{\mathrm{iL}}\, i(t,x) \ , \\ &a(t) = \sqrt{R_0 t / C_0} \big/ R_{\mathrm{iL}} \ , \ b(t,x) = x/2 \cdot \sqrt{R_0 C_0 / t} \ , \end{aligned} \tag{24}$$

with erfc as a complementary error function. To ensure a validity of the above equations at the TL of a finite length l, it is terminated by a time-dependent resistance $r_{\mathrm{iR}}(t)$ matching it perfectly in the time interval of interest. The relative errors evaluated in case of $K = J = 256$ are shown in Fig. 4 for all methods.

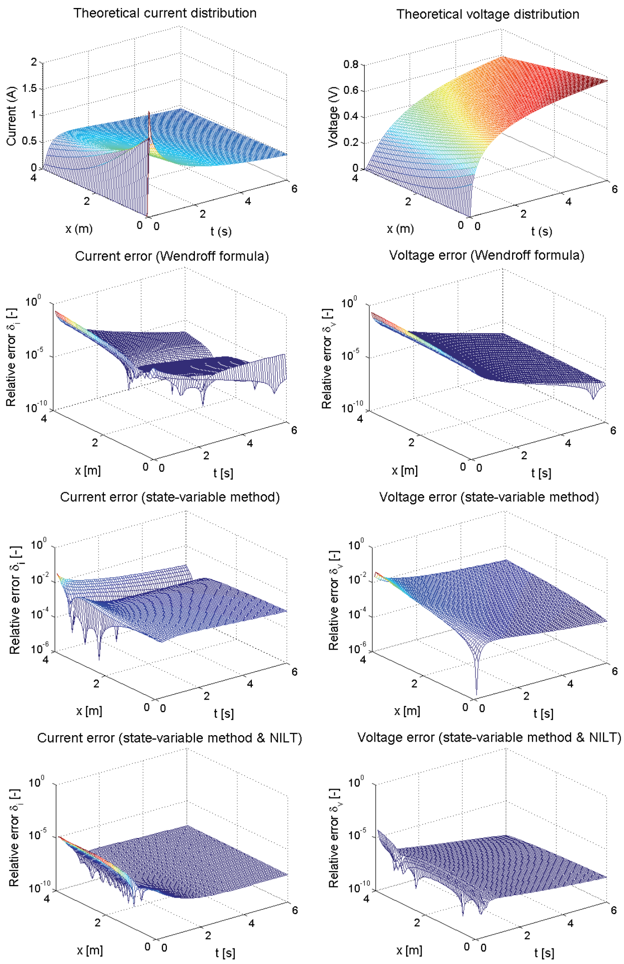

Fig. 4. Thomson cable relative errors (Wendroff vs. state variables).

Param.	$f(t,x)$	Semirelative sensitivity $S_\gamma = \gamma \dfrac{\partial f(t,x)}{\partial \gamma}$
$\gamma \equiv R_0$	$i(t,x)$	$-\underline{1}(t)\dfrac{e^{-b^2(t,x)}}{R_{\mathrm{iL}}\sqrt{\pi}}\left[a(t)+b(t,x)\right]+a(t)\left[a(t)+2b(t,x)\right]i(t,x)$
	$v(t,x)$	$\underline{1}(t)\dfrac{e^{-b^2(t,x)}}{\sqrt{\pi}}a(t)-R_{\mathrm{iL}}a(t)\left[a(t)+2b(t,x)\right]i(t,x)$
$\gamma \equiv C_0$	$i(t,x)$	$\underline{1}(t)\dfrac{e^{-b^2(t,x)}}{R_{\mathrm{iL}}\sqrt{\pi}}\left[a(t)-b(t,x)\right]-a^2(t)i(t,x)$
	$v(t,x)$	$-\underline{1}(t)\dfrac{e^{-b^2(t,x)}}{\sqrt{\pi}}a(t)+R_{\mathrm{iL}}a^2(t)i(t,x)$
$\gamma \equiv R_{\mathrm{iL}}$	$i(t,x)$	$\underline{1}(t)\dfrac{2e^{-b^2(t,x)}}{R_{\mathrm{iL}}\sqrt{\pi}}a(t)-\left(2a(t)\left[a(t)+b(t,x)\right]+1\right)i(t,x)$
	$v(t,x)$	$-\underline{1}(t)\dfrac{2e^{-b^2(t,x)}}{\sqrt{\pi}}a(t)+2R_{\mathrm{iL}}a(t)\left[a(t)+b(t,x)\right]i(t,x)$

Tab. 4. Semirelative sensitivities for Thomson cable.

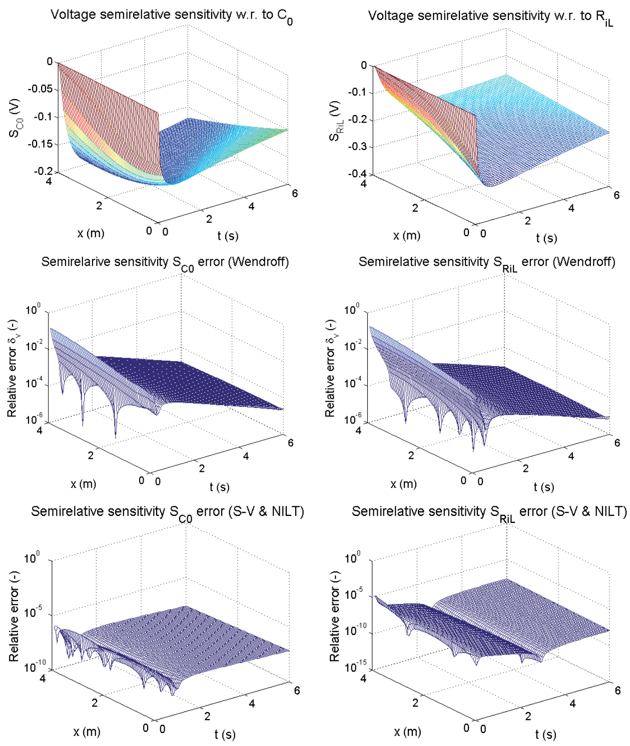

Fig. 5. Thomson cable semirelative sensitivities and relative errors.

The semirelative sensitivities based on (24) have been derived, see Tab. 4, and some graphical results are shown in Fig. 5. Relative errors increase generally at beginning of a time interval where discontinuities occur. In this special case, when a terminating resistance $r_{\mathrm{iR}}(t)$ is time dependent, as defined in Fig. 3, the matrix **A** in (7) has to be evaluated repeatedly in each time step with the help of (24), for $x = l$ substituted, when the Wendroff method is used. Similarly, in case of the state-variable method in the time-domain, **P** in (15) must repeatedly be determined. For the s-domain solution (22), $Z_{\mathrm{iR}}(s) = \sqrt{R_0/sC_0}$ is used to match the load.

5. Examples & CPU Time Evaluation

The following examples will process a simple (2+1)-conductor transmission line system in Fig. 6. The Thévenin internal resistance matrices and voltage vectors are

$$\begin{aligned} &\mathbf{R}_{\mathrm{iL}} = \begin{bmatrix} R_{\mathrm{iL1}} & 0 \\ 0 & R_{\mathrm{iL2}} \end{bmatrix}, \ \mathbf{v}_{\mathrm{iL}}(t) = \begin{bmatrix} v_{\mathrm{iL1}}(t) \\ 0 \end{bmatrix} \\ &\mathbf{R}_{\mathrm{iR}} = \begin{bmatrix} R_{\mathrm{iR1}} & 0 \\ 0 & R_{\mathrm{iR2}} \end{bmatrix}, \ \mathbf{v}_{\mathrm{iR}}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{25}$$

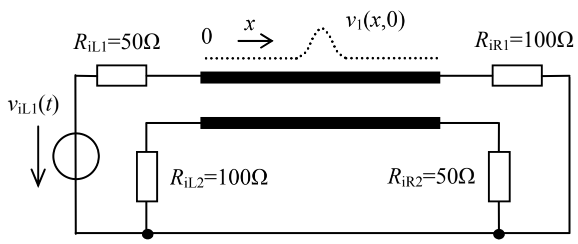

Fig. 6. Simple (2+1)-conductor transmission line system.

First, the MTL is considered under zero initial conditions, i.e. $\mathbf{v}(x,0) = \mathbf{0}$ and $\mathbf{i}(x,0) = \mathbf{0}$, while $v_{iL1}(t) = \sin^2(\pi t/2\cdot 10^{-9})$, $0 \le t \le 2\cdot 10^{-9}$, and $v_{iL1}(t) = 0$, otherwise. The MTL's length $l = 0.4$ m and $\mathbf{P}_0(x) \in \{\mathbf{R}_0(x), \mathbf{L}_0(x), \mathbf{G}_0(x), \mathbf{C}_0(x)\}$ is any of its p.-u.-l. matrix defined as

$$\mathbf{P}_0(x) = \mathbf{P}_0 e^{px} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} e^{px} \tag{26}$$

with the individual primary parameters $R_{11} = R_{22} = 0.1$ Ω/m, $R_{12} = 0.02$ Ω/m, $L_{11} = L_{22} = 494.6$ nH/m, $L_{12} = 63.3$ nH/m, $G_{11} = G_{22} = 0.1$ S/m, $G_{12} = -0.01$ S/m, $C_{11} = C_{22} = 62.8$ pF/m, and $C_{12} = -4.9$ pF/m [18]. If $p = 0$, for all the matrices, an MTL becomes uniform. An inhomogeneity is introduced by the value $p = \ln(2)/l \approx 1.733$ to get two-time greater p.-u.-l. parameters at the MTL's end compared to its beginning. The examples for both uniform and nonuniform MTLs are shown in Fig. 7.

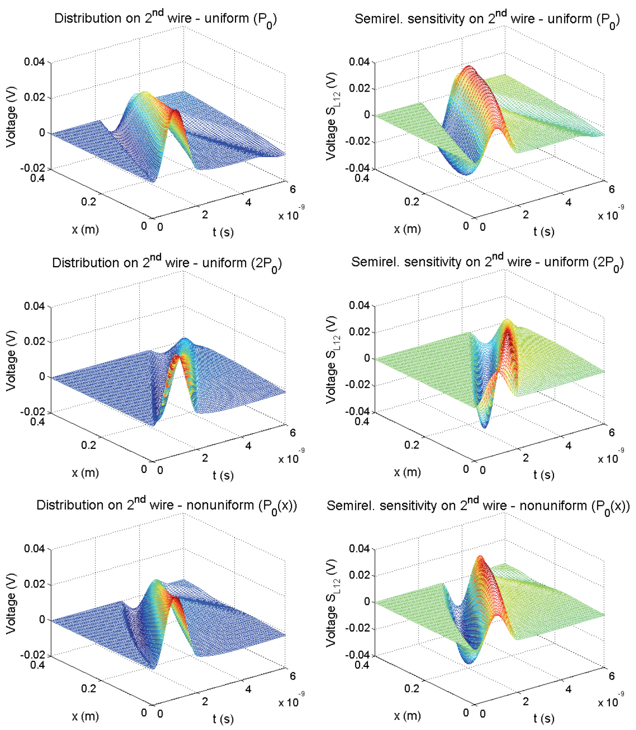

Fig. 7. Voltage distributions (left) and their semirelative sensitivities with respect to $L_{12} \in \mathbf{L}_0(x)$ (right) for uniform and nonuniform MTL.

It can be seen when doubling all p.-u.-l. parameters ($2\mathbf{P}_0$ in Fig. 7) the MTL time delay increases, roughly twice (as reactive parameters are increased). In case of a nonuniform MTL, a wave velocity is not constant, but it decreases gradually along the line leading to intermediate time delay.

The methods can easily be adapted to get responses to MTL initial voltage or current distributions. The examples in case of the voltage distribution on the first wire $v_1(x,0) = \sin^2(\pi(4x/l - 3/2))$, if $3l/8 \le x \le 5l/8$, or $v_1(x,0) = 0$, otherwise, with all the other quantities equal zero, are shown in Fig. 8. All the results for linear MTLs were also compared with a Laplace transform method [12], with very good matching.

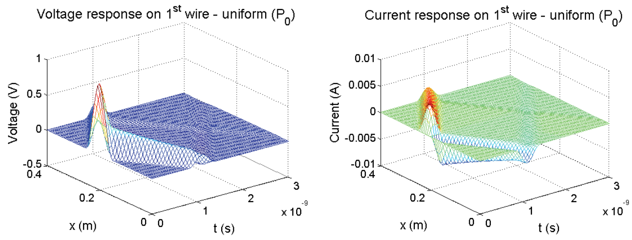

Fig. 8. Voltage (left) and current (right) responses to unitial $v_1(x,0)$.

In the end, some experiments with the simulation of nonlinear MTLs via the Wendroff method are shown in Fig. 9. In this case, the equation (7) is modified on a form $\mathbf{x}^j = (\mathbf{A}^{j-1})^{-1}(\mathbf{B}^{j-1}\mathbf{x}^{j-1}+\mathbf{D}^j)$, when $\mathbf{A}^{j-1}$ and $\mathbf{B}^{j-1}$ depend on live $\mathbf{x}^{j-1}$ and must be evaluated repeatedly in each time step. The nonlinearity is introduced through $\mathbf{C}_0$ matrix, in which $C_{11} = C_{22}$ p.-u.-l. capacitances are replaced by the voltage-dependent ones, namely $C(v_i) = C_{ii}/(1+|v_i|/V_p)^2$, $i = 1,2$, for $V_p = 0.75$ V. The MTL excitation corresponds to that used for the example in Fig. 7.

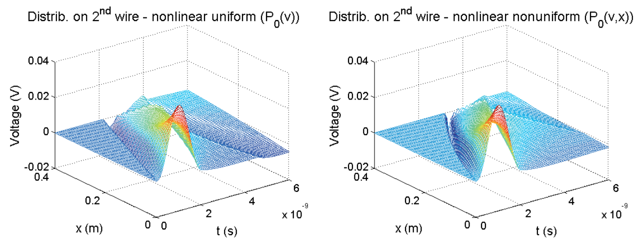

Fig. 9. Voltage distributions for nonlinear uniform/non-uniform MTL.

To evaluate efficiencies of the above techniques the CPU times have been assessed for increasing numbers of grid points (the Wendroff method) and MTL sections (the s-domain state-variable method and NILT), as a function of the numbers of MTL's wires, see Fig. 10 and 11.

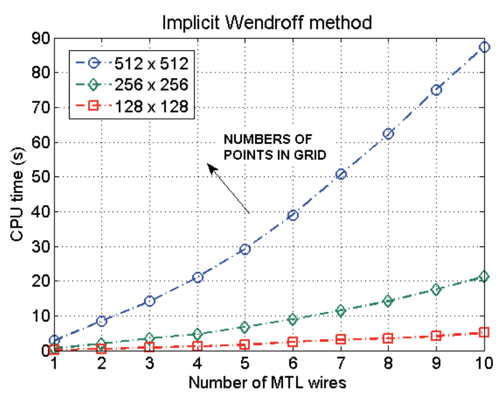

Fig. 10. CPU times for implicit Wendroff method in Matlab.

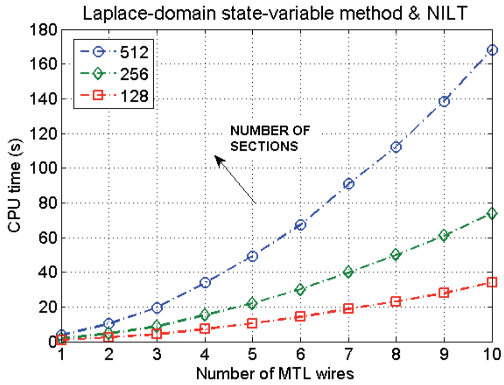

Fig. 11. CPU times for *s*-domain state-variable method in Matlab.

6. Conclusions

All the computer experiments have been performed on a PC 2GHz/2GB, in the Matlab language. It seems from an error analysis the Laplace-domain state-variable method connected with an NILT technique is the most accurate one, at least when discontinuities occur. It will depend, of course, on the type of the NILT. The direct time-domain approaches seem to be comparable, but the state-variable technique depends on a method of a matrix exponential function evaluation which can rather be a problem for high-order matrices. Besides, both the discussed state-variable approaches are only intended for linear MTLs, although its time-domain formulation can be processed by more general case taking also a non-linearity into account. The implicit Wendroff method is simpler to be used for nonlinear MTL cases, and based on the first experiments, just this direction will further be investigated. As follows from CPU times in Fig. 10 and 11, the implicit Wendroff method seems to be the fastest. The time-domain state-variable method depends strongly on a matrix exponential evaluation process, and is markedly more time and RAM consuming. Based e.g. on a Taylor series and taking 256 sections, a CPU time reached up to 370 s for a 10-wire TL, and RAM did not allow more sections. Further works are planned in future to analyze the error levels vs. discretization parameters in both methods.

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