Modeling Deterministic Chaos Using Electronic Circuits

Jiří PETRŽELA, Zdeněk HRUBOŠ, Tomáš GOTTHANS

Dept. of Radio Electronics, Brno University of Technology, Purkyňova 118, 612 00 Brno, Czech Republic

petrzelj@feec.vutbr.cz

Abstract. *This paper brings a note on systematic circuit synthesis methods for modeling the dynamical systems given by mathematical model. Both classical synthesis and integrator based method is demonstrated via the relatively complicated real physical systems with possible chaotic solution. A variety of the different active building blocks are utilized to make the final circuits as simple as possible while preserving easily measurable voltage-mode state variables. Brief experimental verification, i.e. oscilloscope screenshots, is presented. The observed attractors have some structural stability and good relationship to their numerically integrated counterparts.*

Keywords

Autonomous system, state attractor, chaos, analog oscillator, dynamical motion.

1. Introduction

Chaos can be roughly considered as the long-term unpredictable dynamical behavior and resembles noise in many aspects. It is a rare type of motion observable in the autonomous nonlinear dynamical systems with at least three degrees of freedom. The typical features of the chaotic signals are its sensitivity to the tiny changes of the initial conditions and broad-band continuous frequency spectrum. This can be useful in many practical circumstances like securing communication channels [1], masking signals [2], spreading data sequence or for a generation of the random numbers [3]. Of course, chaos is an unwanted phenomenon in many situations. The essential problem which is faced here is how to distinguish such motion from long transient behavior. A spectrum of the so-called Lyapunov exponents (LE) is often used as a quantifier [4] of chaotic motion. These real numbers measure the average ratio of the exponential separation of the two neighborhood trajectories. For chaos it is necessary to have one positive LE, the second one represents a direction of the flow and must converge to zero. The last one must be negative with the largest absolute value since the flow is dissipative.

Another useful tool for chaos visualization is one-dimensional bifurcation diagram (BD). As some important system parameter varies this diagram is composed of the cascaded set of Poincare sections [5]. For the sufficiently high resolution graph it is necessary to use very small parameter step as well as to numerically integrate the state space trajectory for the time long enough. Chaos is up to date also from the theoretical standpoint because it is universal phenomenon repeatedly reported from many distinct scientific fields [6]. Overall analysis usually begins by handling with the dimensionless differential equations thus a physical interpretation of the individual state variables or the system parameters are unimportant. All that matters is preserving a global behavior as well as the attracting sets of the given mathematical model.

2. Mathematical Models

To prove and show the universality of the proposed circuit synthesis methods several mathematical models have been considered. Assume that the first dynamical system represents the task taken from the Newtonian dynamics. Its state variables are position, velocity and acceleration. Such motion can be described by a single third-order differential equation in the form

$$\psi(x) = \dddot{x} - \varphi_1 \ddot{x} + \varphi_2 \dot{x} - \varphi_3 x \tag{1}$$

where φ_i are constant parameters adopted from paper [7], namely $-\varphi_1 = \varphi_2 = -\varphi_3 = 0.8$. It is evident that the state volume shrinking ratio is given by $\varphi_1 = -0.8$. For simplicity assume the nonlinear function is quadratic

$$\psi(x) = \pm x^2 , \tag{2}$$

or more complex function composed of the finite number of the constant segments

$$\psi(x) = \alpha_0 + \sum_{i=1}^{N} \alpha_i \operatorname{sign}(x - \beta_i) \tag{3}$$

where α_i are the stair heights and β_i determine the positions of the vector field boundary planes. The total number of the fixed points corresponds to N and the local vector field geometry is formed by the unstable focus with stability index 1.

It has been verified that the first dynamical system can generate the so-called Rossler-type attractor and the second system provides even number of the spirals. The third mathematical model under inspection is a member of extensive class of the dynamical systems with the cyclically symmetrical vector fields. Such systems can be generally expressed using scalar function as

$$\dot{x} = f(x,y,z), \quad \dot{y} = f(y,z,x), \quad \dot{z} = f(z,x,y).. \quad (4)$$

In particular, the Halvorsen´s attractor [8] can be observed in the case if defining function is

$$f(x,y,z) = -\sigma x - 4y - 4z - y^2. \quad (5)$$

The flow dissipation is uniquely determined by constant $\sigma = 1.27$ leading to the state space volume contraction ratio $\Delta V = -3.81$. The dynamical system (4) and (5) is known for its strong exponential divergence characterized by the largest LE about 0.7899. Finally, let us consider the model of single inertia excitation/inhibition neuron described by the set of the ordinary differential equations [9]

$$\begin{aligned} &\dot{x} = y + ax^2 - x^3 - z + I, \quad \dot{y} = 1 - dx^2 - y, \\ &\dot{z} = \mu[b(x - x_0) - z]. \end{aligned} \quad (6)$$

Taking into account the basic function of this neuron I is the external injected current and x_0 is the initial potential. To design the universal model of neuron each parameter should be fully adjustable through a variable resistor. For the first, second and third system given above some routing-to-chaos scenario has been experimentally confirmed using one variable parameter (resistor). It turns out that φ_3 and σ is a good choice.

3. Classical Circuit Synthesis

There exist several ways how to realize analog chaotic oscillator. The most of these procedures are straightforward and have been already published in books and journal articles. One possibility is to implement the linear part of the vector field by higher-order admittance function [10]. This is an approach suitable especially for equation (1). To manifest this fact let us assume the third-order general admittance function is in the form of fraction

$$Y(s) = \frac{I(s)}{U(s)} = \frac{a_3 s^3 + a_2 s^2 + a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}. \quad (7)$$

Here $I(s)$ and $U(s)$ are Laplace transforms of input current and voltage respectively. Of course, (7) can be expanded into continuous fraction using repeated division or partial fractions expansion. Since every chaotic system has one or multiple unstable equilibrium represented usually by some negative a_i or b_i this synthesis method leads in general to the necessity of creating the negative resistors, capacitors, inductors or even frequency dependent negative resistors. This proposition does not hold for every chaotic oscillator.

For example, Chua's oscillator [11] has admittance network (7) which is eventually passive. Laplace's operator can be considered in time-domain as derivative of the fundamental network quantity. Let us consider the two-port shown in Fig. 1 and situated on the left side of eventually variable resistor R_3. As the first approximation, assume that general current conveyor (GCC) defined by simple relations

$$V_x = \alpha V_y, \qquad I_y = \beta I_x, \qquad I_z = \gamma I_x \quad (8)$$

is used. For such a case one can get

$$\begin{aligned} a_3 &= c_1 c_2 c_3 r_1 r_2, \\ a_0 &= \frac{1}{r_3}, \\ b_0 &= -\alpha(\beta + \gamma), \\ a_1 &= c_1 + c_2 + c_3(\beta + \gamma + 1)(1 - \alpha), \\ b_2 &= -c_2 c_3 r_1 r_2 \alpha\beta, \\ b_1 &= -\alpha\beta[c_2(r_1 + r_2) + c_3 r_1], \\ a_2 &= c_1[c_2(r_1 + r_2) + c_3 r_1(\gamma + 1)] \\ &\quad + c_2 c_3[r_2(\beta + 1 - \alpha) - \alpha\beta(r_1 + r_2)]. \end{aligned} \quad (9)$$

The next step is based on another assumption that GCC is replaced by the integrated circuit marked as EL2082. It is a negative second generation current conveyor (CCCII-) [12] with current gain β externally controllable by some DC current source. By substitution [13] the associated voltage transfer constant $\alpha = 1$, current transfer constant $\beta = 0$ and current gain $\gamma \to -1$ (which holds if the control voltage of EL2082 equals to 1 V) one can get significant simplification of (9) resulting into the formula for admittance

$$Y(s) = c_1 c_2 c_3 r_1 r_2 s^3 + c_1 c_2 (r_1 + r_2) s^2 + (c_1 + c_2) s + \frac{1}{r_3}. \quad (10)$$

In spite of the fact that γ affects all terms of the third-order admittance it has been experimentally verified that this parameter can be used for smooth tracing of chaos evolution through the well known period-doubling bifurcation sequence. The reason for this lies in the fact that it directly changes the time constant of the last differential equation excluding the nonlinear term. By comparing (10) and values of individual φ_i´s the normalized values of the capacitors and resistors can be considered as

$$c_1 = c_2 = \frac{\varphi_2}{2}, \quad c_3 = \frac{\varphi_2^2}{\varphi_1^2}, \quad r_1 = r_2 = -2\frac{\varphi_1}{\varphi_2^2}. \quad (11)$$

Note that these relations are very simple and this is the major advantage of the proposed circuit. The capacitances and resistances in (11) are supposed to be fixed. By definition the situation $\varphi_1 = 0$ eventually turns (10) into the second-order admittance and motions associated with third-order systems cannot be modeled. This circuit cannot be used for the conservative dynamics either. Strictly speaking, it is quite difficult to realize volume preserving systems, but it is still possible, see [14] for further details.

The circuitry implementation of (1) together with (2) is shown in Fig. 1. For this conception an integrated circuit AD633 has been used as a four-quadrant multiplier. It has five high impedance inputs and versatile transfer function $W = K(X1\text{-}X2)(Y1\text{-}Y2) + Z$ with internally trimmed constant

$K = 0.1$. A nonlinear element input current equals to $I_{in}=(K/r_4)V_{in}^2$ and the normalized values of the resistors are $r_3 = -1/\varphi_3$ and $r_4=K/\xi$. Here ξ is a parameter of the nonlinear function. It is possible to generate a mirror chaotic attractor simply by connecting node X1 to the input signal and X2 to the ground. The list of the passive element values calculated using terms (11) is $C_1 = C_2 = 5$ nF, $C_3 = 40$ nF, $R_1 = R_2 = 10$ kΩ, $R_3 = 12.5$ kΩ and $R_4 = 1$ kΩ. Resistor R_3 is supposed to be variable over finite range to allow some routing-to-chaos scenario, as it is displayed in high resolution (parameter step $\Delta\varphi_3 = 0.0005$) in Fig. 1. The piece of the extensive gallery of the oscilloscope screenshots is shown in Fig. 4.

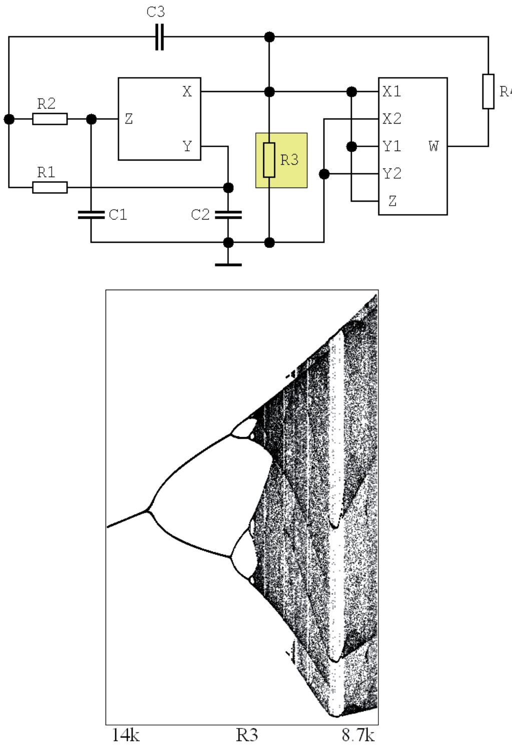

Fig. 1. Circuitry implementation of quadratic canonical oscillator and the corresponding BD with a resistor adopted as a bifurcation parameter.

The linear admittance network in Fig. 1 can be also used as a core engine for generation of the multi-spiral attractors. From fundamental nullor definition, the CCII block can be directly replaced by standard voltage feedback operational amplifier. To design a one-port with the desired AV curve (3), parallel connection of the comparators can be used as it is demonstrated in Fig. 3. Each comparator has its own voltage threshold level β_i (it directly specifies one boundary plane) derived from the resistor divider. This feature allows us to study the effect of AV curve non-symmetry on the global behavior of the circuit without changing feeding DC voltage sources. Each comparator switches its output voltage between positive and negative saturation.

These voltages contribute current to the summing node and the final current is eventually conveyed to make a voltage drop on a single resistor. Normally we should deal with the individual state space segments separately in order to obtain the values of the conversion resistors. This leads to the necessity of solving the non-homogenous system of the linear algebraic equations. Fortunately the AV characteristics we are looking for is odd-symmetrical so that each conversion resistor has the same normalized value $r_i = 2V_{sat}/\zeta$ where ζ is the basic step level. For the values of circuit components given above $\zeta = 200$ μA. If the oscillator is required to operate in the lower frequency band the hysteresis effect of TL084 does not have negative effect on AV curve. For practical verification of the measured multi-spiral chaotic attractors the passive circuit components of the linear admittance network are $R_1 = R_2 = 2.5$ kΩ, $R_3 = 1$ kΩ, $C_1 = C_2 = 33$ nF and $C_3 = 100$ nF. For the nonlinear resistor values $R_{d1} = R_{d6} = 11$ kΩ and $R_{d2} = R_{d3} = R_{d4} = R_{d5} = 2$ kΩ should be utilized if symmetrical voltage supply ±15 V is considered. Up to six spirals have been experimentally confirmed, as it is demonstrated in Fig. 6. The associated AV curves of the nonlinear resistor are given in Fig. 5. For the purpose of current summation and current-to-voltage conversion AD844 integrated circuit is an optimal choice. There are several possible modifications of the multi-spiral oscillator. For example, it should be noted that if the input admittance (7) with (10) employs CCII+ with $\gamma = 1$ the nonlinear AV characteristic can be composed of the segments with only positive slopes. Such curve can be realized by diodes. By using current-mode approach with negligible effect of the parasitic capacitances due to the low nodal impedances there is also chance to improve oscillator performance in the frequency domain.

4. Integrator Based Synthesis

The universal approach to obtain an electronic circuit suitable for modeling complicated system of the differential equations including multi-grid attractors [15] is based on the integrator block schematic. Three basic building blocks are necessary: inverting integrator, differential or summing amplifier and some two-port with a desired transfer curve. The main disadvantage is the large amount of active and passive circuit elements. This drawback is even more significant if only voltage-mode active devices (probably voltage feedback amplifiers) are used. It is because one active block is needed for each signals summation or inversion. On the contrary to this, current-mode (CM) summation is automatically done using a single node. Moreover, better frequency responses of CM devices reduce their filtering effect which is highly unwanted. In other words, a limited gain-bandwidth can destruct the state space attractor. The key elements are lossless integrators with positive second generation current conveyor (CCII+) with output voltage follower, which is commercially available as AD844 [16]. More details about this method can be found in [17].

In the case of (4) the easiest way for desired nonlinear functions realization is to use four-quadrant voltage-mode analog multiplier, for example AD633 again [18]. It is obvious that by setting $V_{X2} = V_{Y1} = V_Z = 0$ V and connecting node X1 with Y2 we get the desired square rooting two-port cell. In general, the dynamical systems with cyclically symmetrical vector field are difficult to realize due to the necessity of three complex nonlinear feedback functions. The advantage of (4) is that feedback branches can be directly summed. Although the linear transformation of the coordinates can make the linear part of the vector field more suitable for practical realization the nonlinear function can turn into much more complicated simultaneously. That is the reason why the basic forms of the equations are used for the circuit design. To validate integrator-based circuit design approach the dynamical system (4) with (5) has been utilized. The final circuitry implementation using only six active devices is provided in Fig. 3 and measured results are in Fig. 7. In this analog oscillator parameter σ is represented by resistors R_1, R_5, R_9. To obtain the typical shape of the state space attractor for Halvorsen´s dynamical system the following list of the linear circuit components has been adopted $R_1 = R_5 = R_9 = 8.1$ kΩ, $R_2 = R_3 = R_6 = R_7 = R_{10} = R_{11} = 2.7$ kΩ and $C_1 = C_2 = C_3 = 2.2$ nF. The individual state variables are easily accessible on the voltage outputs of AD844. Due to chosen large time constant $\tau = 2.2 \cdot 10^{-5}$ s the parasitic properties of the active devices need not to be respected. Circuits mentioned so far have been realized on the contactless board and fed by ±15 V supply voltage.

A very interesting and for future research promising mathematical description to be implemented is a model of single inertia neuron. The authors believe that by coupling many such neurons into layers using fully analog weighting n-ports the real-time analog optimization can be simulated. For this purpose it is necessary to design circuit capable to precise model the neuron itself (6) with all its parameters fully and independently adjustable. It is a three dimensional dynamical system with three nonlinear terms. Thus its circuitry implementation given in Fig. 3 consists of three inverting integrators and amplifiers with TL084 and four analog multipliers (AD633 can be employed). In practice the DC sources are replaced by voltage dividers realized by the potentiometers.

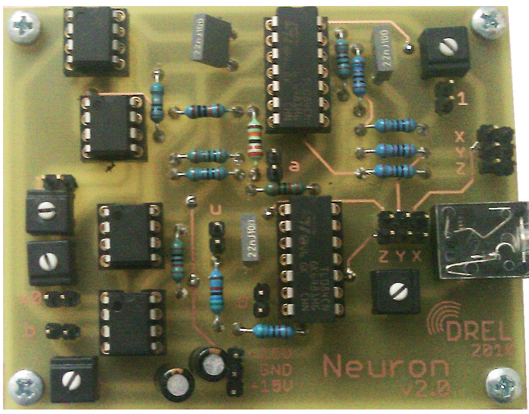

Fig. 2. Prototype of single neuron with audio output.

In the given schematic voltage $V_1 = 1$ V represents constant term in (6), voltage V_2 corresponds to parameter I divided by 100, voltage V_3 and V_4 defines parameter b and x_0 respectively. Other system parameters are defined by gains, namely parameter a by resistor R_{10} and parameter d by resistor R_{14}. Time constant of circuit is determined by the capacitors $C_1 = C_2 = C_3 = 100$ nF as well as the associated resistors $R_1 = R_2 = R_3 = R_4 = R_5 = 1$ kΩ, $R_7 = 100$ kΩ and $R_8 = 10$ Ω. The main drawback of the proposed circuit lies in the necessity of many integrated circuits. The experimental results are in Fig. 8.

5. Conclusion

The advantage of the inductorless structure of the first chaotic oscillator proposed in this paper is an easy relation between parameters of the mathematical model (equivalent eigenvalues) and the circuit elements. Moreover, a huge number of the laboratory experiments prove a perfect agreement between numerical integration and practical measurement, see Fig. 9 and Fig. 10. This is the reason why Pspice circuit simulator does not bring new information and particular simulations in time domain have been performed but the results are not given. The multi-spiral oscillator can be also considered as canonical in the sense of minimum circuit components. The second oscillator proves the versatility of the circuit synthesis based on the integrator block schematic. The Halvorsen´s equations have been picked up as a challenge to practically implement oscillator with cyclically symmetrical vector field. To date, referring to the best knowledge of the authors the experimental observation of the corresponding Halvorsen´s attractor has not been published. The same proposition holds for the model of neuron. The practical application of the synthesized circuit is still an unanswered question and represents an interesting topic for further research. The rather extensive list of the references provides an opportunity to study chaotic motion from many different viewpoints. The algebraically simple dynamical systems with minimum terms [19] are suitable for practical training in the analog chaotic oscillator design. Some useful procedures leading to the chaos generation can be also found in [20]. A quite different overview on the chaos evolution is given in [21].

Acknowledgements

The research is a part of the COST action IC 0803, which is financially supported by the Czech Ministry of Education under grant no. OC09016. The first author would also like to thank the Grant Agency of the Czech Republic for their support through project number 102/09/P217. The research leading to these results has received funding from the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. 230126.

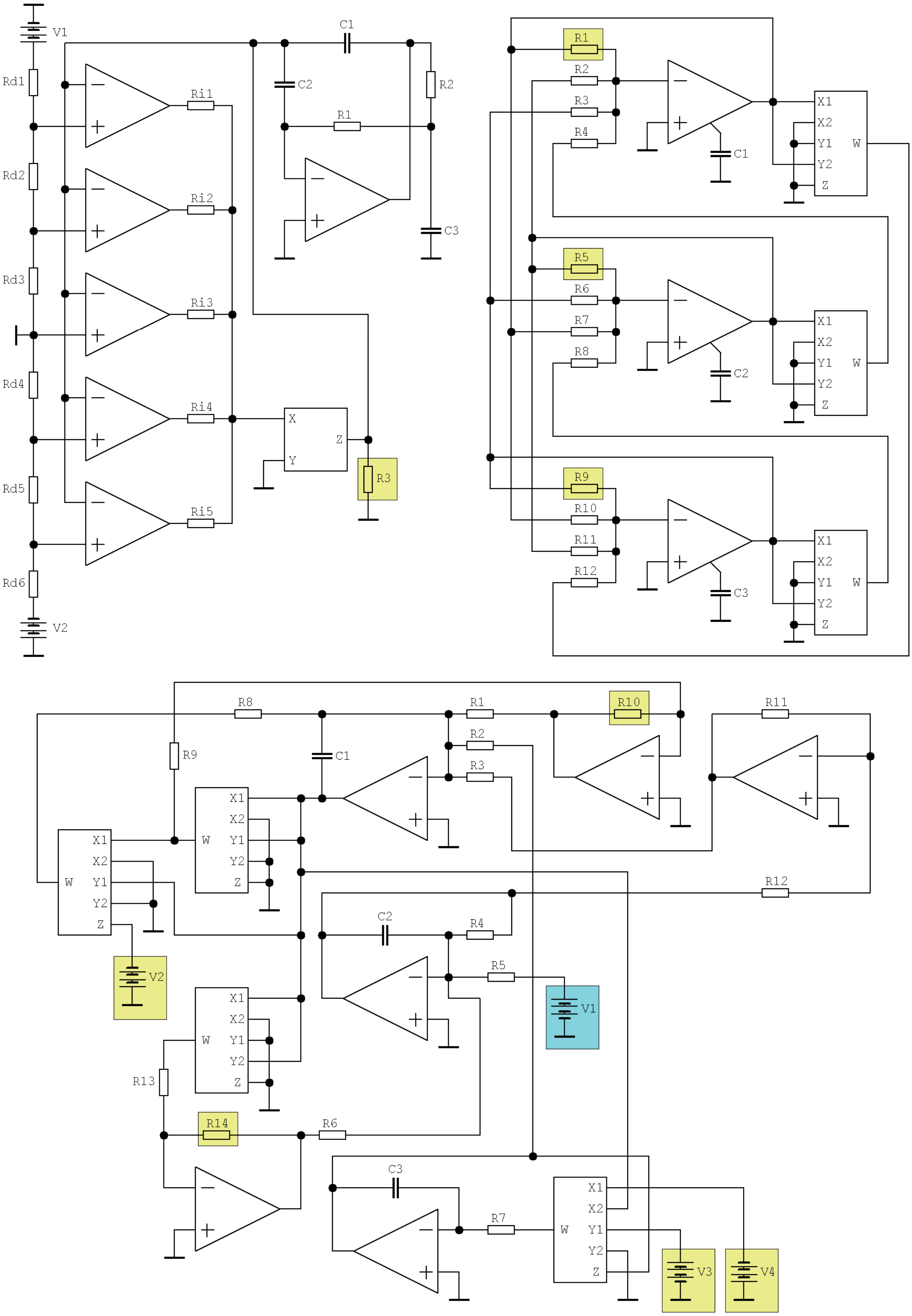

Fig. 3. Circuitry implementation of the one-dimensional multi-spiral generator (upper left picture), Halvorsen´s cyclically symmetrical dynamical system (upper right picture), and fully analog representation of single inertia neuron (lower picture).

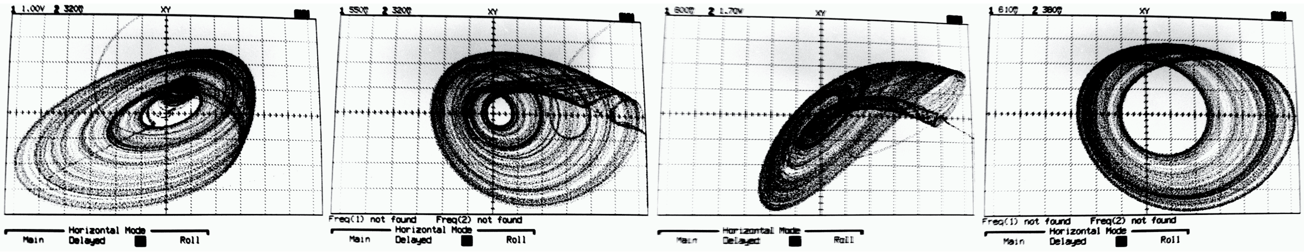

Fig. 4. Selected plane projections of the signals generated by quadratic oscillator and measured by means of digital oscilloscope HP54603B.

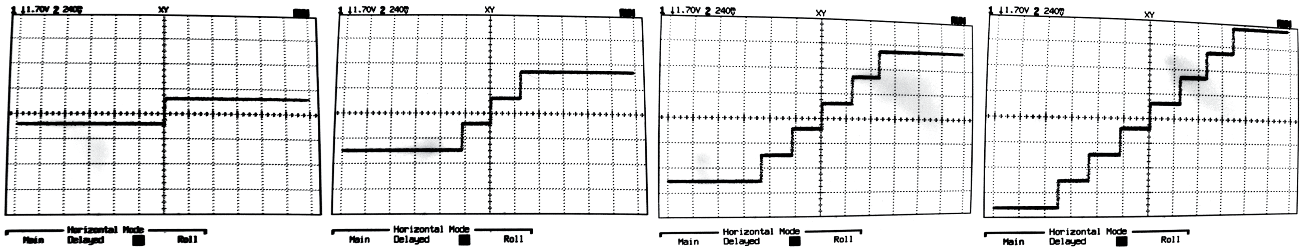

Fig. 5. The piecewise-constant AV curve of the nonlinear resistor used inside a generator of multi-spiral attractors measured using AD844 connected as current-to-voltage converter, digital oscilloscope HP54603B.

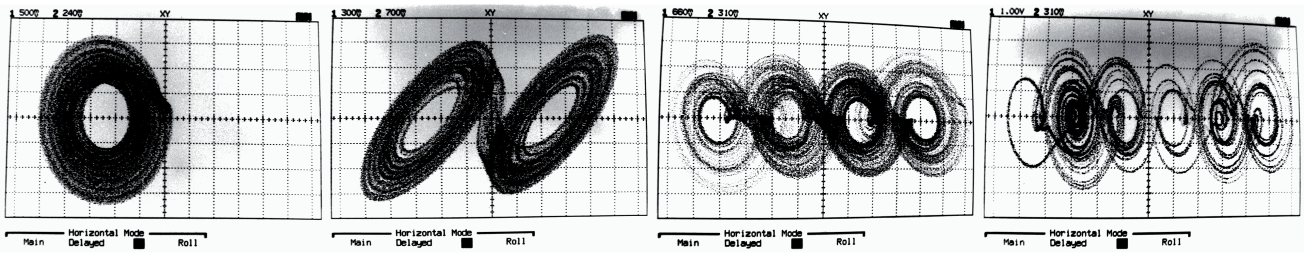

Fig. 6. Generation of multi-spiral attractors using canonical admittance network and stair-type of the nonlinear resistor, selected screenshots obtained using digital oscilloscope HP54603B.

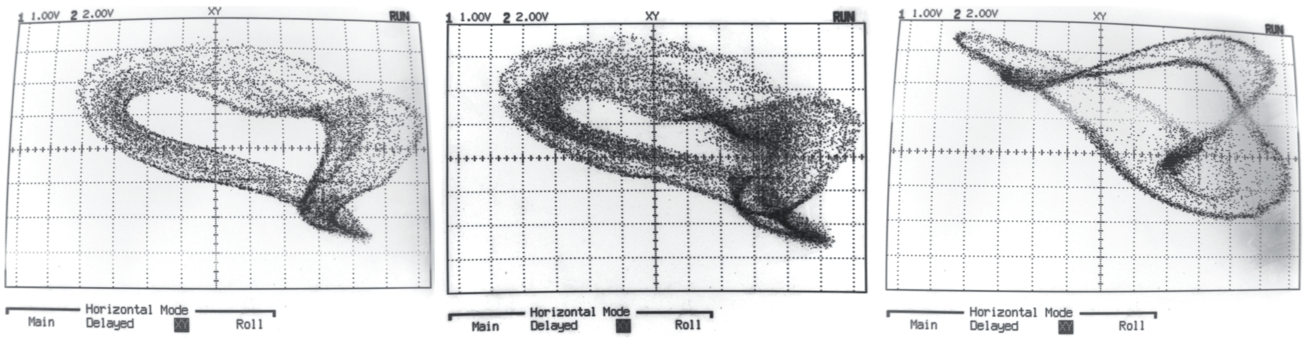

Fig. 7. Typical plane projections of the cyclically symmetrical Halvorsen´s oscillator measured using digital oscilloscope HP54603B, chaos evolution through changing the parameter σ.

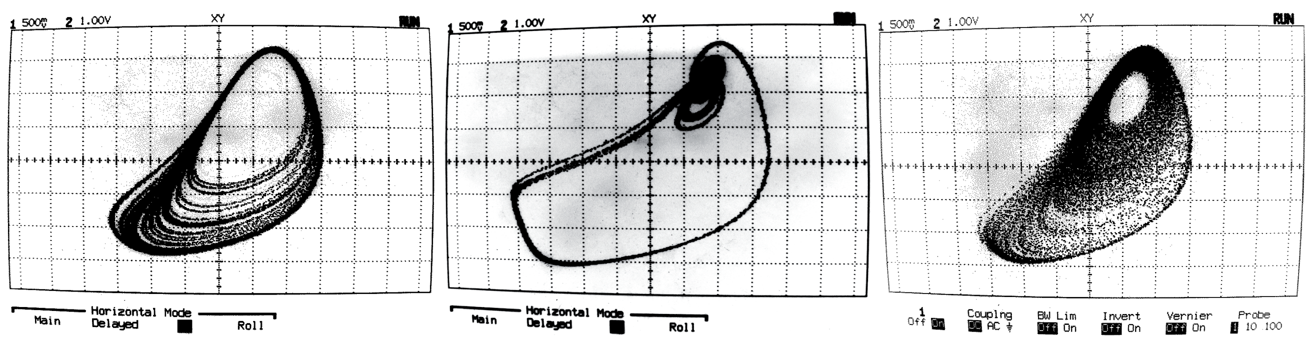

Fig. 8. Typical plane projections and associated waveforms measured using digital oscilloscope HP54603B, Halvorsen´s oscillator.

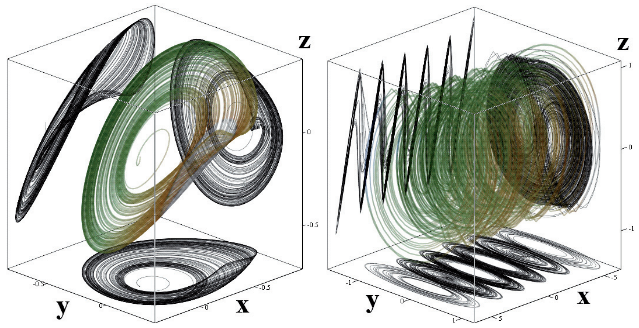

Fig. 9. Numerical integration process, typical chaotic attractors for oscillator with quadratic nonlinearity and multi-spiral generator.

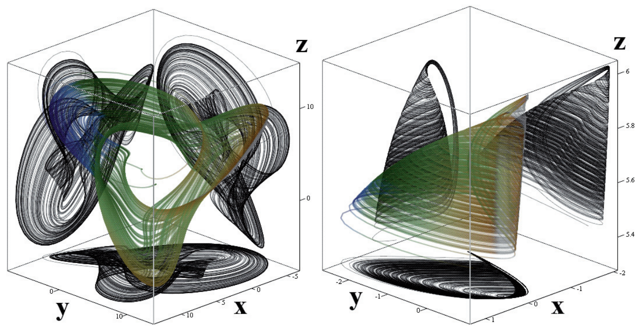

Fig. 10. Numerical integration process, typical chaotic attractors for Halvorsen´s dynamical system and third-order model of neuron.

References

[1] LI, Z., SOH, J., WEN, CH. *Switched and Impulsive Systems*. Berlin: Springer, 2005.

[2] GALAJDA, P., KOCUR, D. Chua´s circuit in spread spectrum communication systems. *Radioengineering*, 2002, vol. 11, no. 2, p. 6 – 10.

[3] ADDABBO, T., FORT, A., ROCCHI, S., VIGNOLI, V. *Intelligent Computing Based on Chaos*. Berlin: Springer, 2009.

[4] GRYGIEL, K., SZLACHETKA, P. Lyapunov exponents analysis of autonomous and nonautonomous set of ordinary differential equations. *Acta Physica Polonica B,* 1995, vol. 26, no. 8, p. 1321 to 1331.

[5] PETRŽELA, J. *Modeling of the Strange Behavior in the Selected Nonlinear Dynamical Systems, Part II: Analysis*. Brno: Vutium Press, 2010.

[6] CHEN, G., UETA, T. *Chaos in Circuits and Systems*. London: World Scientific, 2002.

[7] PETRŽELA, J., HANUS, S., KOLKA, Z. Simple chaotic oscillator: from mathematical model to practical experiment. *Radioengineering*, 2006, vol. 15, no. 1, p. 6 – 11.

[8] SPROTT, J. C. *Chaos and Time-Series Analysis*. Oxford University Press, 2003.

[9] CAMINITI, R., GHAZIRI, H., GALUSKE, R., HOF, P. R., INNOCENTI, G. M. Evolution amplified processing with temporally dispersed slow neuronal connectivity in primates. *Proceedings of the National Academy of Sciences*, 2009, vol. 106, no. 46, p. 19551 – 19556.

[10] WYK, M. A., STEEB, W. H. *Chaos in Electronics*. Kluwer Academic Publishers, 1997.

[11] ITOH, M. Synthesis of electronic circuits for simulating nonlinear dynamics. *International Journal of Bifurcation and Chaos,* 2001, vol. 11, no. 3, p. 605 – 653.

[12] ČAJKA, J., DOSTÁL, T., VRBA, K. General view on current conveyors. *International Journal of Circuit Theory and Applications*, 2004, vol. 32, no. 3, p. 133 – 138.

[13] Current Mode Multiplier EL2082 Datasheet, Elantec, 1996.

[14] PETRŽELA, J., SLEZÁK, J. Conservative chaos generators with CCII+ based on the mathematical model of nonlinear oscillator. *Radioengineering*, 2008, vol. 17, no. 3, p. 19 – 24.

[15] LU, J., YU, S., LEUNG, H., CHEN, G. Experimental verification of multidirectional multi-scroll chaotic attractors. *IEEE Transaction on Circuit and Systems I*, 2006, vol. 53, no. 1, p. 149 – 165.

[16] High Speed Monolithic Operational Amplifier AD844 Datasheet, Analog Devices, 2003.

[17] PETRŽELA, J. Integrator-based synthesis of the selected chaotic oscillators. *Slaboproudý obzor*, 2009, vol. 4, no. 3, p. 9 – 12 (in Czech).

[18] Low Cost Analog Multiplier AD633 Datasheet, Analog Devices, 2002.

[19] SPROTT, J. C. Simple chaotic systems and circuits. *American Journal Physics*, 2000, vol. 68, no. 8, p. 758 – 763.

[20] PETRŽELA, J. *Modeling of the Strange Behavior in the Selected Nonlinear Dynamical Systems, Part I: Oscillators*. Brno: Vutium Press, 2008.

[21] ŠPÁNY, V., GALAJDA, P., GUZAN, M., PIVKA, L., OLEJÁR, M. Chua´s singularities: great miracle in circuit theory. *International Journal of Bifurcation and Chaos*, 2010, vol. 20, no. 10, p. 2993 – 3006.

About Authors...

Jiří PETRŽELA was born in Brno, Czech Republic, in 1978. He received the MSc. and PhD. degrees from the Brno University of Technology in 2003 and 2007 respectively. His research interest covers the nonlinear dynamics, chaos theory and analog circuit design. Currently he is an assistant professor at the Department of Radio Electronics.

Zdeněk HRUBOŠ was born in Uherske Hradiste, Czech Republic, in 1984. He received his MSc. degree in 2009 from the University of Technology, Brno. He is currently studying his Ph.D. study at the same university. His research is aimed in the computer analysis and synthesis of electronics circuits and nonlinear systems.

Tomáš GOTTHANS was born in Brno, Czech Republic, in 1985. He received the MSc. degree from the Brno University of Technology in 2010. Now he is a Ph.D. student at the Department of Radio Electronics, Brno University of Technology. His research interests include nonlinear dynamics, microprocessors, mixed-mode circuit synthesis.