Linear Phase Second Order Recursive Digital Integrators and Differentiators

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Abstract. *In this paper, design of linear phase second order recursive digital integrators and differentiators is discussed. New second order digital integrators have been designed by using Genetic Algorithm (GA) optimization method. Thereafter, by modifying the transfer function of these integrators appropriately, new digital differentiators have been obtained. The proposed digital integrators and differentiators accurately approximate the ideal ones and have linear phase response over almost entire Nyquist frequency range. The proposed operators also outperform the existing operators in terms of both magnitude and phase response.*

Keywords

Linear optimization, Genetic Algorithm optimization, digital integrator, digital differentiator.

1. Introduction

The frequency response of an ideal digital integrator is $1/j\omega$ and that of differentiator is $j\omega$, where $j=\sqrt{-1}$ and ω is the angular frequency in radians.

Al-Alaoui [1] has proposed interpolation method to design digital integrator. This interpolation method has become very popular. Papamarkos-Chamzas [2] has used linear programming optimization method to propose the design of digital integrators. Hsu-Wang-Yu [3] has designed integrators by using genetic algorithm method. Ngo [4] has used Newton-Cotes integration method to design digital integrators. Tseng-Lee [5] has used fractional delay to design digital integrators. Gupta-Jain-Kumar (GJK) [6-8] have also proposed digital integrators by using interpolation method. Al-Alaoui [9] has also designed a family of digital integrators and differentiators by using interpolation and simulated annealing optimization method. Varshney-Gupta-Visweswaran [10] have also used interpolation method to design new digital differentiator, they then designed half order digital integrator and differentiator by using it.

Recently, pole-zero optimization method has been used by Upadhyay-Singh (US) [11] for the design of integrator and differentiator with absolute relative error (ARE) $\leq 0.48\%$ over $0 \leq \omega \leq 0.94\pi$ radians. The maximum phase deviation of the US [11] differentiator is 24.5^{o} from the ideal linear phase response and integrator has nearly linear phase response for almost entire Nyquist frequency range except near $\omega = 0$ radians.

In this paper, Genetic Algorithm optimization method [12-13] is used to design the proposed digital integrators. Digital differentiators have been designed by modifying the transfer function of the proposed integrators by using the approach described in [14]. The proposed (integrator-I and differentiator-I) and (integrator-II and differentiator-II) have absolute relative error (ARE) $\leq$ 0.20% over $0 \leq \omega \leq 0.80\pi$ radians and 0.30% over $0 \leq \omega \leq 0.95\pi$ radians, respectively. It has also been shown in results that proposed integrators and differentiators have almost linear phase response for the entire Nyquist frequency range including $\omega = 0$ radians.

These proposed second order digital integrators and differentiators can be used in almost all engineering disciplines including control, communications, biomedical, radar and signal processing applications [15–20].

This paper is organized as follows. Section 2 presents the brief about genetic algorithm. Section 3 describes the designing of integrators using GA and designing of differentiators by using the transfer function of designed integrators. In Section 4 comparisons of the designed integrators and differentiators with the existing ones are carried out. The conclusions are given in Section 5.

2. Use of GA in Design of Digital Integrators

The flow chart of Genetic algorithms (GA) optimization method [12-13] is shown in Fig. 1. First of all a second order transfer function is assumed as an integrator, then absolute relative error of this integrator with respect to ideal integrator is defined as a fitness function. Initial values of coefficients were decided. A set of possible solutions (individuals) is generated randomly from within a pre-defined range, they are represented as binary strings.

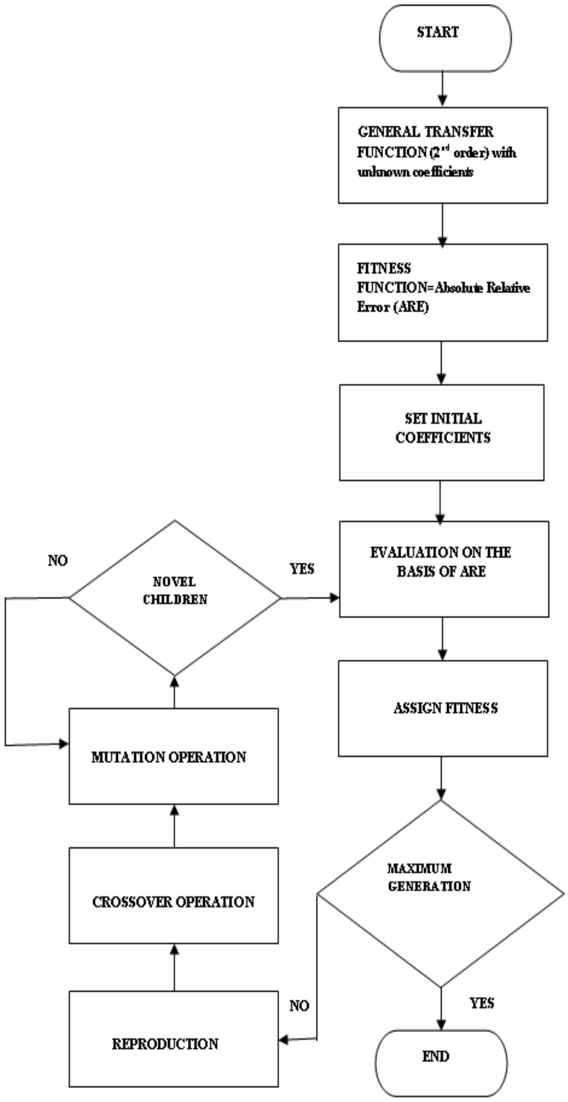

Fig. 1. Flow chart of Genetic Algorithm (GA) optimization method.

Then fitness function is applied on them and on the basis of their performance a fitness value is given to each individual. The reproduction operator is used to privilege good individuals and remove bad ones. The population size has been kept constant while creating new population.

Two individuals are selected on the basis of their fitness value (higher fitness value has higher chance for selection). These individuals are known as parents. In the next step, crossover and mutation processes has been applied over parents to form new individuals (children).

The main idea of crossover is that the children should be better than their parents. Crossover can be classified as one-point, two-point and uniform crossover. Here, one point crossover is used on parents to generate children. For this, a point is chosen which is known as crossover point and the segments to the right of this point are exchanged.

Let us have two parents solution as

x1=111010111000111 and x2=101110110011001

The crossover point is chosen between bit 8 and 9 (leftmost bit is assumed as bit 1).

x1=11101011¦1000111 and x2=10111011¦0011001

Then, their children are

y1=111010110011001 and y2=101110111000111

The crossover is always defined with crossover rate (R_C). Population size defines the number of individuals in one generation. If the population size is small, then GA has fewer possibilities to perform crossover and only a small part of search space can be explored. On the other hand, if the population size is large, then GA slows down. Let the population size be N, that means there are N individuals in each generation. In each generation $N*R_C$ individuals will perform crossover. If the crossover probability is high, then children will quickly add to the population. If the crossover probability is too high then high-performance individuals will be discarded very quickly before selection can produce improvements and if the crossover probability is low than it may stagnate the search due to loss of exploration power. In mutation a few randomly chosen bits of a chromosome will switch from 1 to 0 or from 0 to 1. Mutation process is used in GA to avoid local optimization. For example, lets x1=100010101011011 and the mutational bit is bit 7, then the child is y1=100010001011011. Mutation is also defined with mutation rate (R_M), which is a probability by which each bit position of each individual in the intermediate population undergoes a random change. If L is the length of the chromosome then R_M*N*L number of mutations will occur per generation. If mutation probability is too high then GA will become a random search.

Crossover and mutation processes are repeated until maximum number of generations has reached. During the entire algorithm, the all time best solution is stored and returned at the end of algorithm.

3. Design of Digital Integrators and Differentiators

3.1 Design of Digital Integrators

When dealing with an integrator and differentiator not only the amplitude but also the phase information is important. However, obtaining the efficient amplitude and phase response of any integrator or differentiator is difficult. Therefore, a very strong research effort is focused on the design of new digital integrator and differentiator with efficient frequency (both magnitude and phase) response. Here, classical binary encoding Genetic Algorithm has been implemented in design of digital integrators.

First of all, a recursive second order transfer function with unknown coefficients is considered as a digital integrator whose coefficients have to be optimized using GA.

$$I(z) = \frac{T\left(a_1 z^2 + a_2 z + a_3\right)}{\left(z^2 + b_1 z + b_2\right)} . \tag{1}$$

The set of possible solutions (individuals) forms the population, which is evolved by means of the selection, crossover and mutation operators. Each individual (chromosome) consists of 20 bits. Here 50 generations with the population size of 20 individuals have been used, therefore, the maximum number of fitness evaluation per iteration is 1000. In order to create a new generation of individuals, crossover and mutation operators have to be applied. In each generation, the individuals are decoded and evaluated according to a fitness function. Here, absolute relative error (*ARE*) of the integrator $I(z)$ as compared to ideal one is taken as a fitness function. It is defined as:

$$ARE = \left| \frac{I_{\text{int}}(\omega) - I\left(e^{j\omega t}\right)}{I_{\text{int}}(\omega)} \right| \tag{2}$$

where, $I_{\text{int}}(\omega)$ is the transfer function of ideal digital integrator (defined earlier) and $I(e^{j\omega T})$ is calculated by replacing z by $e^{j\omega T}$ in (1). Here sampling period of the filter T is assumed as 1 second in the frequency plots and the Nyquist frequency is π radians/ sample.

Initially, the coefficients are assumed as a_1= +0.0001, a_2= +0.0001, a_3= +0.0001, b_1= +0.0001 and b_1= +0.0001.

Crossover and mutation operators have been used to create new population. The main idea of crossover is that the children should be better than their parents. Crossover can be classified as one-point, two-point and uniform crossover. Here, one point crossover is used on parents to generate children. For this, a point is chosen which is known as crossover point and the segments to the right of this point are exchanged, crossover rate (R_C) has been set to 0.60.

In mutation a few randomly chosen bits of a chromosome will switch from 1 to 0 or from 0 to 1. Mutation process is used in GA to avoid local optimization, mutation rate (R_M) has been set to 0.10. The better individual (the solution with lower ARE) is chosen and used to create children. Before each fitness values computation, the chromosome is compared with the previous ones. If it was similar, fitness values are copied instead of repeating the already performed computation.

Tab. 1 describes all the GA parameters used to obtained best results in this paper. The whole optimization process is performed in MATLAB 7.

Using GA, twelve integrators are obtained which have less ARE compare to the ideal one. All the coefficients of these twelve designed integrators are shown in Tab. 2 and their ARE response is shown in Fig. 2. It is seen that all these integrators have less ARE but design XI and XII have minimum ARE in the entire Nyquist frequency range. Thus these are called as proposed integrator-I and II in this paper.

S.No	Parameters	Value
1.	Initial coefficients	(0.0001,0.0001,0.0001, 0.0001, 0.0001)
2.	No. of bits used in digital representation	20
3.	Fitness function	Absolute Relative Error (ARE) of I(z) (Eq.2)
4.	Mutation	Uniform
5.	Population size	20
6.	Crossover rate	0.60
7.	Mutation rate	0.10
8.	Maximum generation	50
9.	Total time of calculation	

Tab. 1. GA parameters.

S.No	a(1)	a(2)	a(3)	b(1)	b(2)
I.	+0.0858	+0.9146	+0.5116	-0.4891	-0.5108
II.	+0.8648	+0.6562	+0.0611	-0.4183	-0.5818
III.	+0.0875	+0.9099	+0.5107	-0.4863	-0.4999
IV.	+0.0999	+0.9269	+0.4999	-0.4799	-0.5199
V.	+0.0934	+0.9242	+0.5065	-0.4901	-0.5207
VI.	-0.8641	-0.5856	-0.0518	-0.5044	-0.4899
VII.	+1.7467	+1.1706	+0.1022	+1.0146	-2.0070
VIII.	+0.0909	+0.9175	+0.5064	-0.4817	-0.5080
IX.	+0.0878	+0.9161	+0.5065	-0.4880	-0.5107
X.	+0.0845	+0.9151	+0.5114	-0.4893	-0.5108
XI.	+0.0868	+0.9148	+0.5122	-0.4881	-0.5107
XII.	+0.8647	+0.5998	+0.0541	-0.4812	-0.5142

Tab. 2. Coefficients of designed integrators.

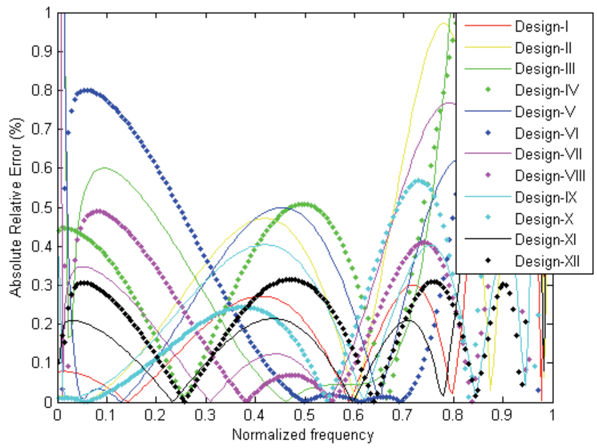

Fig. 2. Percentage relative error response of all twelve designed integrators

The transfer functions of the proposed integrator-I and II using the coefficients mentioned in Tab. 2 (Design XI and XII).

$$I_1(z)=\frac{T\left(0.0868z^2+0.9148z+0.5122\right)}{\left(z^2-0.4881z-0.5107\right)}, \tag{3}$$

$$I_2(z)=\frac{T\left(0.8647z^2+0.5998z+0.0541\right)}{\left(z^2-0.4812z-0.5142\right)}. \tag{4}$$

The magnitude response of ideal and proposed integrators is shown in Fig. 3. It can be seen that the magnitude response curves of the proposed and ideal integrators are overlapping each other.

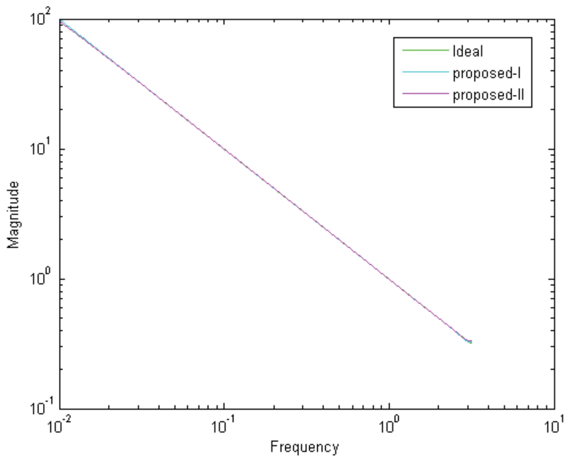

Fig. 3. The magnitude response of the ideal integrator and the proposed integrators; $I_1(z)$ and $I_2(z)$.

3.2 Designing of Digital Differentiators

In this paper digital differentiators have been designed by inverting and modifying the transfer function of the proposed digital integrators (3-4) as suggested by Al-Alaoui [14]. On inverting (3), one pole appears outside the unit circle at z = - 9.946. This unstable pole is replaced by inverting it to get a stable pole at z = -1/9.946. The resulting change in amplitude is compensated by multiplying the denominator by a factor of 9.946. On inverting of (4), all the poles are inside the unit circle, therefore stabilization and compensation are not required. The resulting transfer function $D_1(z)$ and $D_2(z)$ are

$$D_1(z)=\frac{1}{T}\frac{\left(z^2-0.4881z-0.5107\right)}{(0.0868)(9.946)(z+0.1005)(z+0.5933)}, \tag{5}$$

$$D_2(z)=\frac{1}{T}\frac{\left(z^2-0.4812z-0.5142\right)}{\left(0.8647z^2+0.5998z+0.0541\right)}. \tag{6}$$

These are called as proposed differentiator-I and II in this paper. The magnitude response of ideal and proposed differentiators is shown in Fig. 4. It can be seen that the magnitude response curves of the proposed and ideal differentiators are overlapping each other.

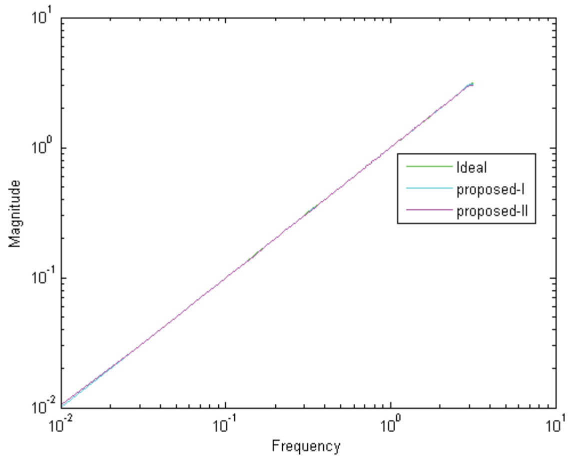

Fig. 4. The magnitude response of ideal differentiator and the proposed differentiators; $D_1(z)$ and $D_2(z)$.

4. Comparison of the Proposed Integrators and Differentiators with the Existing Ones

To define and compare the efficiency of the designed integrators, various recently proposed integrators have been considered. These are, Gupta-Jain-Kumar1 integrator ($I_{GJK1}(z)$) [6], Gupta-Jain-Kumar2 integrator ($I_{GJK2}(z)$) [7] and Upadhyay-Singh integrator ($I_{US}(z)$) [11]. Their transfer functions are

$$I_{GJK1}(z)=\frac{0.34T(z+2.541)\left(z^2-0.2081z+0.03858\right)}{z^2(z-1)}, \tag{7}$$

$$I_{GJK2}(z)=\frac{0.329T(z+2.663)\left(z^2-0.2079z+0.03864\right)}{z^2(z-1)}, \tag{8}$$

$$I_{US}(z)=\frac{0.8657T\left(z^2+0.681z+0.0628\right)}{\left(z^2-0.4975z-0.5025\right)}. \tag{9}$$

The ARE response of the above mentioned integrators are shown in Fig. 5.

It can be seen that the proposed integrators; $I_1(z)$ and $I_2(z)$ (3-4) have ARE $\leq$ 0.20% over $0 \leq \omega \leq 0.80\pi$ radians and $\leq$ 0.30% over $0 \leq \omega \leq 0.95\pi$ radians, respectively, thus these can be regarded as wideband digital integrators. Existing wideband digital integrators Gupta-Jain-Kumar1 [6], Gupta-Jain-Kumar2 [7], and Upadhyay-Singh [11] have ARE $\leq$ 3.0% over $0 \leq \omega \leq \pi$ radians, $\leq$ 2.8% over $0 \leq \omega \leq 0.95\pi$ radians and $\leq$ 0.48% over $0 \leq \omega \leq 0.94\pi$ radians, respectively.

This is verified by the simulation results (Fig. 5) that the proposed wideband integrators outperform all these existing integrators over entire Nyquist frequency range.

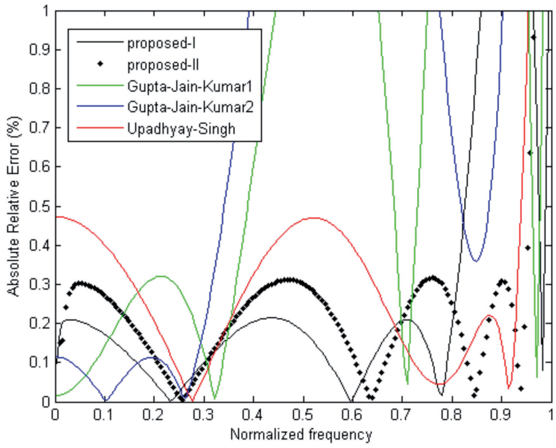

Fig. 5. Percentage relative error response of the proposed integrators; $I_1(z)$ and $I_2(z)$, Gupta-Jain-Kumar integrators [6,7] and Upadhyay-Singh integrator [11].

The efficiency of the designed differentiators has also been shown by comparing with the recently proposed differentiators.

These are Gupta-Jain-Kumar1 differentiator ($D_{GJK1}(z)$) [6], Gupta-Jain-Kumar2 differentiator ($D_{GJK2}(z)$) [7], and Upadhyay-Singh differentiator ($D_{US}(z)$) [11]. Their transfer functions are

$$D_{GJK1}(z)= \frac{z^2(z-1)}{T(0.34)(2.541)(z+0.3935)(z^2-0.2081z+0.03858)} \tag{10}$$

$$D_{GJK2}(z)= \frac{z^2(z-1)}{T(0.329)(2.663)(z+0.3755)(z^2-0.2079z+0.03864)} \tag{11}$$

$$D_{US}(z)=\frac{0.5805(z^2+0.99z-1.99)}{T(z^2+0.681z+0.0628)} \tag{12}$$

The ARE response of the above mentioned differentiators is shown in Fig. 6.

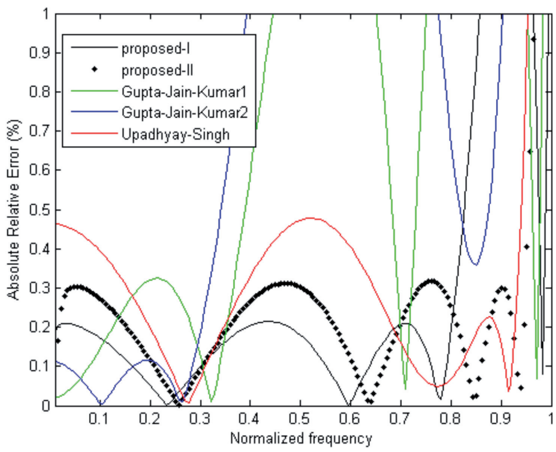

Fig. 6. Percentage relative error response of proposed differentiators; $D_1(z)$ and $D_2(z)$, Gupta-Jain-Kumar differentiators [6,7] and Upadhyay-Singh differentiator [11].

It can be seen that the proposed differentiators; $D_1(z)$ and $D_2(z)$ have ARE $\leq 0.20\%$ over $0 \leq \omega \leq 0.80\pi$ radians and $\leq 0.30\%$ over $0 \leq \omega \leq 0.95\pi$ radians, respectively, thus these can be regarded as wideband digital differentiators. Existing digital differentiators Gupta-Jain-Kumar1 [6], Gupta-Jain-Kumar2 [7], and Upadhyay-Singh [11] have ARE $\leq 3.0\%$ over $0 \leq \omega \leq \pi$ radians, $\leq 2.8\%$ over $0 \leq \omega \leq 0.95\pi$ radians and $\leq 0.48\%$ over $0 \leq \omega \leq 0.94\pi$ radians, respectively. This is verified by the simulation results (Fig. 6) that the proposed wideband differentiators outperform all these existing differentiators over entire Nyquist frequency range.

Phase response of the proposed integrators and differentiators are shown in Fig. 7.

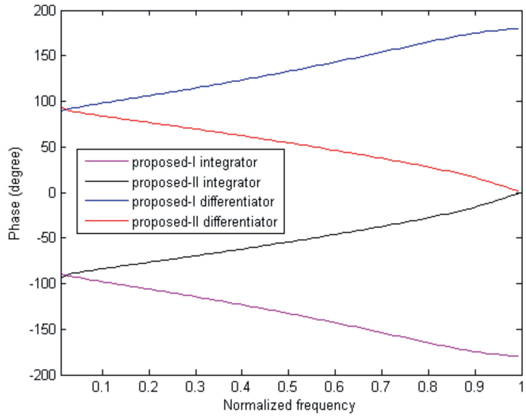

Fig. 7. Phase response of proposed integrators; $I_1(z)$, $I_2(z)$, and differentiators; $D_1(z)$, $D_2(z)$.

The maximum phase deviation from ideal linear phase response in case of Gupta-Jain-Kumar1 integrator and differentiator [6] is 35.3° and 12°, respectively, Gupta-Jain-Kumar2 integrator and differentiator [7] is 34.2° and 11°, respectively, and Upadhyay-Singh differentiator [11] is 24.5°; whereas Upadhyay-Singh integrator [11] has nearly linear phase response for almost entire Nyquist frequency range except near ω =0 radians. However, it can be seen (Fig. 7) that the proposed operators have linear phase response over almost entire Nyquist frequency range.

It is verified by the simulation results (Fig. 5- 7) that the proposed wideband operators outperform all the existing operators in frequency domain analysis (magnitude and phase response).

5. Conclusion

The proposed integrators and differentiators accurately approximate the ideal ones with very small absolute relative error and linear phase response over almost entire Nyquist frequency range. It is verified by the simulation results that the proposed second order wideband operators outperform all the existing operators in frequency domain analysis. The low order (second order), less relative error and linear phase response of the proposed integrators and differentiators make them useful in real time applications.

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