CRLBs for Pilot-Aided Channel Estimation in OFDM System under Gaussian and Non-Gaussian Mixed Noise

Jiyan HUANG[1], *Peng WANG*[2], *Xiaofeng SHEN*[1]

[1] Dept. of Electronic Engineering, University of Electronic Science and Technology of China, Chengdu, China
[2] School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore

huangjiyan@uestc.edu.cn, epwang@ntu.edu.sg, sxf.uestc@163.com

Abstract. *The determination of Cramer-Rao lower bound (CRLB) as an optimality criterion for the problem of channel estimation in wireless communication is a very important issue. Several CRLBs on channel estimation have been derived for Gaussian noise. However, a practical channel is affected by not only Gaussian background noise but also non-Gaussian noise such as impulsive interference. This paper derives the deterministic and stochastic CRLBs for Gaussian and non-Gaussian mixed noise. Due to the use of the non-parametric kernel method to build the PDF of non-Gaussian noise, the proposed CRLBs are suitable for practical channel environments with various noise distributions.*

Keywords

Channel estimation, orthogonal frequency-division multiplexing (OFDM), Cramer-Rao lower bound (CRLB), mixed noise.

1. Introduction

To overcome multipath fading in wireless communication, the orthogonal frequency-division multiplexing (OFDM) method was proposed for high-bit-rate communications [1]. Channel state information (CSI) is very important for the OFDM to achieve optimal diversity combination and coherent detection at the receiving end. In the absence of CSI, pilot-assisted channel estimators can be used to estimate CSI [1]-[5].

Although many channel estimators have been proposed, their performances were seldom studied. The papers [6], [7] designed the optimal pilot pattern for the OFDM system. The methods in [8], [9] were proposed for analyzing the channel estimation techniques in OFDM system. These methods [8], [9] based on variance analysis cannot determine the best achievable accuracy for the channel estimators. Cramer-Rao lower bound (CRLB) sets a lower limit for the covariance matrix of any unbiased estimate of parameters and determines the physical impossibility of the variance of an unbiased estimator being less than the bound [10], [11]. The deterministic and stochastic CRLBs are defined for deterministic unknown process and random Gaussian process, respectively. The deterministic CRLB on channel estimation was derived for the uniform Gaussian noise case [12]. The CRLB for Multi-inputs Multi-outputs channel estimation was reported in [13]. It should be noted that the performance analyses in [8-9, 12-13] were based on Gaussian noise assumption. However, a practical channel is affected by not only Gaussian background noise but also non-Gaussian noise such as impulsive interference [3] which is a Gaussian and non-Gaussian mixed noise. Several CRLBs for non-Gaussian noise have been proposed for array processing problem [14], [15]. The CRLB derived for Direction-of-arrival (DOA) estimation in [14] is based on a special noise distribution known as Class A distribution. Since kernel density estimators asymptotically converge to any probability density function (PDF), the non-parametric kernel method is an attractive and powerful tool to estimate PDF of a non-Gaussian noise from survey data and has been successfully applied in the performance analysis of array processing problem in non-Gaussian noise environment [15]. Despite the previous works [14], [15], the determination of CRLB as an optimality criterion for the problem of channel estimation in non-Gaussian noise environment is still an opening issue. Since the actual distribution of non-Gaussian noise changes with the channel environment, it is desired to develop a CRLB on channel estimation for non-Gaussian noise with arbitrary distributions. Inspired by [14] and [15], this paper derives the deterministic and stochastic CRLBs on channel estimation in OFDM system for Gaussian and non-Gaussian mixed noise which have not been studied. Due to the use of the non-parametric kernel method to build the PDF of non-Gaussian noise, the proposed CRLBs are suitable for various noise distributions.

The following notational conventions are used in this paper: $\mathbf{A}^T$ is the transpose of matrix $\mathbf{A}$; $\mathbf{A}^H$ is the conjugate transpose of matrix $\mathbf{A}$; $\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}$; $\bar{\mathbf{A}}$ is the real part of matrix $\mathbf{A}$; $\tilde{\mathbf{A}}$ is the imaginary part of matrix $\mathbf{A}$; $tr\{\mathbf{A}\}$ is the trace of matrix $\mathbf{A}$; A_{ij} is the i , j element of matrix $\mathbf{A}$.

2. System Description

Consider an OFDM system with N subcarriers. Without loss of generality in eliminating inter-symbol interference, the length of cyclic prefix is assumed to be longer than the maximum delay spread of wireless channel.

For pilot-assisted channel estimation, the total of N_p pilot symbols is inserted into the N subcarriers at the given locations. The vector of the received frequency-domain signals $\mathbf{Y} = \left[Y(i_1) \ \cdots \ Y(i_{N_p}) \right]^T$ at the pilot locations $\left\{ i_k; 1 \le k \le N_p \right\}$ is:

$$\mathbf{Y} = \mathbf{XH} + \mathbf{v} = \mathbf{Dh} + \mathbf{v} = \mathbf{XFh} + \mathbf{n} + \mathbf{b} \tag{1}$$

where $\mathbf{X} = diag\left\{ \left[X_1 \ \cdots \ X_{N_p} \right] \right\}$ represents the transmitted signals at the pilot locations, $\mathbf{h} = \left[h_1 \ \cdots \ h_L \right]^T$ is the channel impulse response (CIR) with L multipath components, $\mathbf{D} = \mathbf{XF}$, $\mathbf{n} = \left[n_1 \ \cdots \ n_{N_p} \right]^T$ represents the Gaussian background noises, $\mathbf{b} = \left[b_1 \ \cdots \ b_{N_p} \right]^T$ represents the non-Gaussian noise, and $\mathbf{v} = \mathbf{b} + \mathbf{n} = \left[v_1 \ \cdots \ v_{N_p} \right]^T$. Non-Gaussian noise $\mathbf{b}$ is a major factor affecting the accuracy of channel estimation and is assumed to 0 in most studies to simplify the analysis. Both $\mathbf{n}$ and $\mathbf{b}$ are considered in this paper. $\mathbf{F}$ is a $N_p \times L$ matrix with entries:

$$F_{kn} = e^{-j2\pi i_k n/N}, 1 \le k \le N_p, 0 \le n \le L-1. \tag{2}$$

The objective of the channel estimation is to estimate $\mathbf{h}$ from the observation of $\mathbf{Y}$.

CRLB is very important for parameter estimation because it provides a benchmark to evaluate the performance of any unbiased estimator.

For a complex vector $\mathbf{h} = \left[\bar{h}_1 + j\tilde{h}_1 \ \cdots \ \bar{h}_L + j\tilde{h}_L \right]^T$, the unknown parameters can be written as $\boldsymbol{\theta} = \left[\bar{h}_1 \ \cdots \ \bar{h}_L \ \tilde{h}_1 \ \cdots \ \tilde{h}_L \right]^T$.

CRLB is defined as [10], [11]:

$$CRLB(\boldsymbol{\theta}) = tr\left\{ \mathbf{J}_{\boldsymbol{\theta}}^{-1} \right\} \tag{3}$$

Fisher information matrix (FIM) $\mathbf{J}_{\boldsymbol{\theta}}$ is [10], [11]:

$$\mathbf{J}_{\boldsymbol{\theta}} = E\left[\frac{\partial \ln f(\mathbf{Y};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \cdot \left(\frac{\partial \ln f(\mathbf{Y};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T \right] \tag{4}$$

where $f(\mathbf{Y};\boldsymbol{\theta})$ is the PDF.

3. CRLB for Gaussian and Non-Gaussian Mixed Noise

The CRLBs based on the assumption of zero mean Gaussian noise are not suitable for a practical environment where $\mathbf{b}$ is subject to different distributions. The deterministic and stochastic CRLBs considering both Gaussian and non-Gaussian cases for practical channel have to be derived.

For a complex vector $\mathbf{b}$, $f(\mathbf{b})$ can be written as:

$$f(\mathbf{b}) = f(\bar{\mathbf{b}}) \cdot f(\tilde{\mathbf{b}}) \tag{5}$$

Equation (5) assumes that the real and imaginary parts of $\mathbf{b}$ are independent. This assumption is widely used in the literature [16], [17]. There are parametric and non-parametric methods for evaluating $f(\mathbf{b})$. The parametric method can only be used for specific noise distribution such as Gaussian distribution. The non-parametric method is an attractive tool for density estimation of non-Gaussian noise and has been widely used for many applications in color image processing, mobile location, and regression problems [18].

The non-parametric method is developed here to derive the CRLB on channel estimation for practical channel environments because it can be applicable for any noise distribution. The basic procedure of the non-parametric estimation is to create an approximated PDF from a given set of survey measurements. Assume that the survey sets are available for b_i in each pilot channel. For the i th pilot channel, the survey set with size M is $\left\{ \bar{b}_{i1} + j\tilde{b}_{i1} \ \cdots \ \bar{b}_{iM} + j\tilde{b}_{iM} \right\}$. The estimated PDF of $\bar{b}_i$ can be obtained using non-parametric Gaussian kernel method [18]:

$$f_{\bar{b}_i}(\bar{b}_i) = \frac{1}{\sqrt{2\pi} M \bar{d}_i} \sum_{j=1}^{M} \exp\left(-\frac{\left(\bar{b}_i - \bar{b}_{ij} \right)^2}{2\bar{d}_i^2} \right) \tag{6}$$

where the smoothing constant $\bar{d}_i$ is the width of the kernel function which is determined using the method in [18]. Many non-parametric estimators such as histogram method, orthogonal series, and other kernel methods can effectively estimate the PDF and have the similar performance. Gaussian kernel method was chosen due to its similarity with the Euclidean distance and also because it gives better smoothing and continuous properties even with a small number of samples [19]. Another reason is that it is easy to integrate and differentiate and can lead to mathematically tractable solution.

Since n_i is a zero-mean Gaussian random variable with variance σ_i^2, the PDF of $\bar{n}_i$ is given by:

$$f_{\bar{n}_i}(\bar{n}_i) = \frac{1}{\sqrt{\pi\sigma_i^2}} \exp\left(-\frac{\bar{n}_i^2}{\sigma_i^2}\right). \quad (7)$$

The PDF of $\bar{v}_i = \bar{b}_i + \bar{n}_i$ is:

$$f_{\bar{v}_i}(\bar{v}_i) = \int_{-\infty}^{+\infty} f_{\bar{b}_i}(x) f_{\bar{n}_i}(\bar{v}_i - x)dx$$

$$= \frac{1}{M\sqrt{2\pi\left(\sigma_i^2/2+\bar{d}_i^2\right)}} \sum_{j=1}^{M} \exp\left(-\frac{\left(\bar{v}_i - \bar{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\bar{d}_i^2\right)}\right) \quad (8)$$

Similarly, the PDF of $\tilde{v}_i$ is:

$$f_{\tilde{v}_i}(\tilde{v}_i) = \frac{1}{M\sqrt{2\pi\left(\sigma_i^2/2+\tilde{d}_i^2\right)}} \sum_{j=1}^{M} \exp\left(-\frac{\left(\tilde{v}_i - \tilde{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\tilde{d}_i^2\right)}\right). \quad (9)$$

The joint PDF of v_i is:

$$f(v_i) = f_{\bar{v}_i}(\bar{v}_i) f_{\tilde{v}_i}(\tilde{v}_i) \quad (10)$$

$$= k_i \sum_{j=1}^{M} \exp\left(-\frac{\left(\bar{v}_i - \bar{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\bar{d}_i^2\right)}\right) \sum_{j=1}^{M} \exp\left(-\frac{\left(\tilde{v}_i - \tilde{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\tilde{d}_i^2\right)}\right)$$

where $k_i = \dfrac{1}{M^2 2\pi\sqrt{\left(\sigma_i^2/2+\bar{d}_i^2\right)\left(\sigma_i^2/2+\tilde{d}_i^2\right)}}$.

For the vector of unknown parameters $\boldsymbol{\theta}$, the PDF of $f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)$ can be obtained by substituting (1) into (10):

$$f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right) = k_i \sum_{j=1}^{M} \exp\left(-\frac{\left(\bar{Y}(i_i) - \sum_{t=1}^{L}\left(\bar{D}_{it}\bar{h}_t - \tilde{D}_{it}\tilde{h}_t\right) - \bar{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\bar{d}_i^2\right)}\right)$$

$$\cdot \sum_{j=1}^{M} \exp\left(-\frac{\left(\tilde{Y}(i_i) - \sum_{t=1}^{L}\left(\bar{D}_{it}\tilde{h}_t + \tilde{D}_{it}\bar{h}_t\right) - \tilde{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\tilde{d}_i^2\right)}\right). \quad (11)$$

where L is the number of multipath components.

The deterministic CRLB assumes $\mathbf{h}$ to be a deterministic unknown process and the received signal of each subcarrier is independent. Thus the joint PDF $f\left(\mathbf{Y};\boldsymbol{\theta}\right)$ becomes:

$$f\left(\mathbf{Y};\boldsymbol{\theta}\right) = f\left(\mathbf{Y}\,|\,\boldsymbol{\theta}\right) = \prod_{i=1}^{N_p} f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right) \quad (12)$$

Substituting (12) into (4), gives:

$$\mathbf{J}_{\boldsymbol{\theta}} = E\left\{\sum_{i=1}^{N_p}\left(\frac{1}{f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)}\right)^2 \begin{bmatrix} J_{11} & \cdots & J_{1(2L)} \\ \vdots & \ddots & \vdots \\ J_{(2L)1} & \cdots & J_{(2L)(2L)} \end{bmatrix}\right\} \quad (13)$$

where $J_{kt} = \dfrac{\partial f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)}{\partial\theta_k}\left(\dfrac{\partial f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)}{\partial\theta_t}\right)$.

Substituting (11) into $\partial f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)/\partial\theta_k$ ($\theta_k = \bar{h}_k$ or $\theta_k = \tilde{h}_k$), gives:

$$\frac{\partial f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)}{\partial\theta_k} = g_i(v_i)\frac{d\bar{v}_i}{d\theta_k} + s_i(v_i)\frac{d\tilde{v}_i}{d\theta_k} \quad (14)$$

where

$$g_i(v_i) = \sum_{j=1}^{M} \exp\left(-\frac{\left(\bar{v}_i - \bar{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\bar{d}_i^2\right)}\right)\cdot\left(-\frac{\bar{v}_i - \bar{b}_{ij}}{\left(\sigma_i^2/2+\bar{d}_i^2\right)}\right)$$

$$\cdot\sum_{j=1}^{M} \exp\left(-\frac{\left(\tilde{v}_i - \tilde{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\tilde{d}_i^2\right)}\right)k_i,$$

$$s_i(v_i) = \sum_{j=1}^{M} \exp\left(-\frac{\left(\bar{v}_i - \bar{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\bar{d}_i^2\right)}\right)\cdot\sum_{j=1}^{M} \exp\left(-\frac{\left(\tilde{v}_i - \tilde{b}_{ij}\right)^2}{2\left(\sigma_i^2/2+\tilde{d}_i^2\right)}\right)$$

$$\left(-\frac{\tilde{v}_i - \tilde{b}_{ij}}{\left(\sigma_i^2/2+\tilde{d}_i^2\right)}\right)k_i$$

$$\frac{\partial\bar{v}_i}{\partial\bar{h}_k} = -\bar{D}_{ik},\; \frac{\partial\tilde{v}_i}{\partial\bar{h}_k} = -\tilde{D}_{ik},\; \frac{\partial\bar{v}_i}{\partial\tilde{h}_k} = \tilde{D}_{ik},\; \frac{\partial\tilde{v}_i}{\partial\tilde{h}_k} = -\bar{D}_{ik}. \quad (15)$$

Substituting (14) into J_{kt}, gives:

$$J_{kt} = g_i(v_i)^2\frac{d\bar{v}_i}{d\theta_k}\frac{d\bar{v}_i}{d\theta_t} + s_i(v_i)^2\frac{d\tilde{v}_i}{d\theta_k}\frac{d\tilde{v}_i}{d\theta_t} \quad (16)$$

$$+g_i(v_i)s_i(v_i)\frac{d\bar{v}_i}{d\theta_k}\frac{d\tilde{v}_i}{d\theta_t} + s_i(v_i)g_i(v_i)\frac{d\tilde{v}_i}{d\theta_k}\frac{d\bar{v}_i}{d\theta_t}.$$

Substituting (16) into (13), gives:

$$\mathbf{J}_{\boldsymbol{\theta}} = \sum_{i=1}^{N_p} E\left[\left(\frac{g_i(v_i)}{f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)}\right)^2\mathbf{S}_{i1} + \left(\frac{s_i(v_i)}{f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)}\right)^2\mathbf{S}_{i2}\right.$$

$$\left.+\frac{g_i(v_i)s_i(v_i)}{f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)^2}\mathbf{S}_{i3} + \frac{g_i(v_i)s_i(v_i)}{f\left(Y(i_i)\,|\,\boldsymbol{\theta}\right)^2}\mathbf{S}_{i4}\right] = \sum_{i=1}^{N_p}\sum_{k=1}^{4} A_{ik}\mathbf{S}_{ik} \quad (17)$$

where

$$\mathbf{S}_{i1} = \begin{bmatrix} \frac{d\bar{v}_i}{d\theta_1}\frac{d\bar{v}_i}{d\theta_1} & \cdots & \frac{d\bar{v}_i}{d\theta_1}\frac{d\bar{v}_i}{d\theta_{2L}} \\ \vdots & \ddots & \vdots \\ \frac{d\bar{v}_i}{d\theta_{2L}}\frac{d\bar{v}_i}{d\theta_1} & \cdots & \frac{d\bar{v}_i}{d\theta_{2L}}\frac{d\bar{v}_i}{d\theta_{2L}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{D}}_i\bar{\mathbf{D}}_i^T & -\bar{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T \\ -\tilde{\mathbf{D}}_i\bar{\mathbf{D}}_i^T & \tilde{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T \end{bmatrix},$$

$$\mathbf{S}_{i2} = \begin{bmatrix} \bar{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T & \bar{\mathbf{D}}_i\bar{\mathbf{D}}_i^T \\ -\bar{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T & -\bar{\mathbf{D}}_i\bar{\mathbf{D}}_i^T \end{bmatrix},\ \mathbf{S}_{i3} = \mathbf{S}_{i2}^T,\ \mathbf{S}_{i4} = \begin{bmatrix} \tilde{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T & \tilde{\mathbf{D}}_i\bar{\mathbf{D}}_i^T \\ \bar{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T & \bar{\mathbf{D}}_i\bar{\mathbf{D}}_i^T \end{bmatrix},$$

$$\bar{\mathbf{D}}_i = \begin{bmatrix} \bar{D}_{i1} & \cdots & \bar{D}_{iL} \end{bmatrix}^T,\ \tilde{\mathbf{D}}_i = \begin{bmatrix} \tilde{D}_{i1} & \cdots & \tilde{D}_{iL} \end{bmatrix}^T,$$

$$A_{i1} = E\left[\left(\frac{g_i(v_i)}{f(Y(i_i)\,|\,\boldsymbol{\theta})}\right)^2\right] = E\left[\left(\frac{g_i(v_i)}{f(v_i)}\right)^2\right]$$

$$= \int_{-\infty}^{+\infty}\left(\frac{g_i(v_i)}{f(v_i)}\right)^2 f(v_i)\,dv_i = \int_{-\infty}^{+\infty}\frac{g_i(v_i)^2}{f(v_i)}dv_i$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{-\infty}\frac{g_i(v_i)^2}{f(v_i)}d\bar{v}_i d\tilde{v}_i,$$

$$A_{i2} = \int_{-\infty}^{+\infty}\int_{-\infty}^{-\infty}\frac{s_i(v_i)^2}{f(v_i)}d\bar{v}_i d\tilde{v}_i,$$

$$A_{i3} = \int_{-\infty}^{+\infty}\int_{-\infty}^{-\infty}\frac{g_i(v_i)s_i(v_i)}{f(v_i)}d\bar{v}_i d\tilde{v}_i,\ A_{i3} = A_{i4} \tag{18}$$

Many numerical methods such as Matlab function “dblquad” can be used to calculate A_{ik}.

Since $g_i(v_i)s_i(v_i)/f(v_i)$ is an odd function, $A_{i3} = A_{i4} = 0$ and the deterministic CRLB is given by:

$$CRLB\_d(\boldsymbol{\theta}) = tr\left\{\left(\sum_{i=1}^{N_p}\sum_{k=1}^{2}A_{ik}\mathbf{S}_{ik}\right)^{-1}\right\} \tag{19}$$

Generally, the real part and imaginary part of $\mathbf{b}$ have the same distribution. In this case, $A_{i1} = A_{i2}$.

Substituting $A_{i1} = A_{i2}$ into (19), gives:

$$CRLB\_d(\boldsymbol{\theta}) = tr\left\{\left(\sum_{i=1}^{N_p}\sum_{k=1}^{2}A_{ik}\mathbf{S}_{ik}\right)^{-1}\right\}$$

$$= tr\left\{\left(\sum_{i=1}^{N_p}A_{i1}\begin{bmatrix} \bar{\mathbf{D}}_i\bar{\mathbf{D}}_i^T + \tilde{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T & -\bar{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T + \tilde{\mathbf{D}}_i\bar{\mathbf{D}}_i^T \\ -\tilde{\mathbf{D}}_i\bar{\mathbf{D}}_i^T + \bar{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T & \tilde{\mathbf{D}}_i\tilde{\mathbf{D}}_i^T + \bar{\mathbf{D}}_i\bar{\mathbf{D}}_i^T \end{bmatrix}\right)^{-1}\right\}$$

$$= tr\left\{\frac{1}{2}\begin{bmatrix} \mathrm{Re}\left\{\left(\mathbf{D}^H\mathbf{B}^{-1}\mathbf{D}\right)^{-1}\right\} & -\mathrm{Im}\left\{\left(\mathbf{D}^H\mathbf{B}^{-1}\mathbf{D}\right)^{-1}\right\} \\ \mathrm{Im}\left\{\left(\mathbf{D}^H\mathbf{B}^{-1}\mathbf{D}\right)^{-1}\right\} & \mathrm{Re}\left\{\left(\mathbf{D}^H\mathbf{B}^{-1}\mathbf{D}\right)^{-1}\right\} \end{bmatrix}\right\}$$

$$= tr\left\{\left(\mathbf{D}^H\mathbf{B}^{-1}\mathbf{D}\right)^{-1}\right\} = tr\left\{\left((\mathbf{XF})^H\mathbf{B}^{-1}\mathbf{XF}\right)^{-1}\right\} \tag{20}$$

where $\mathbf{B} = diag\left\{\begin{bmatrix} \frac{2}{A_{11}} & \cdots & \frac{2}{A_{N_p 1}} \end{bmatrix}\right\}$.

Remark 1: Note that the CRLB in (20) is similar to the deterministic CRLB in Gaussian noise environment [12]. The only difference is $\mathbf{B} = diag\left\{\begin{bmatrix}\sigma_1^2 & \cdots & \sigma_{N_p}^2\end{bmatrix}\right\}$ for the Gaussian case. Thus, $\mathbf{B}$ can be rewritten as:

$$B_{ii} = \begin{cases} \sigma_i^2 & i \in Gaussian \\ \frac{2}{A_{i1}} & i \in mixed \end{cases} \tag{21}$$

where $i \in Gaussian$ is for Gaussian noise channel, otherwise Gaussian and non-Gaussian mixed noise channel. Equations (20) and (21) give the unified CRLB representation for the cases of Gaussian and mixed channel environments.

Remark 2: It should be pointed out that there are two assumptions for the non-parametric method: survey measurements of the non-Gaussian noise must be provided and non-Gaussian noise is a stationary process. Without these assumptions, the problem of the determination of CRLB in non-Gaussian environment will become unsolvable because it is impossible to obtain the PDF of the non-Gaussian noise. Although non-Gaussian noise may be a non-stationary process during a short time in some cases, the proposed CRLBs can still be applied if this noise is a stationary process for a long time.

Remark 3: There are virtually no simple parametric models for non-Gaussian noise in all channel environments because its PDF changes with channel environments. This implies that it is impossible to propose a parametric CRLB for practical wireless channel. Thus, the proposed CRLBs based on survey measurements and non-parametric methods, which are applicable for all noise distributions, are necessary.

Remark 4: Note that the derived CRLB can also be used for the case that non-Gaussian noise cannot be isolated from Gaussian noise by substituting $\sigma_i^2 = 0$ into A_{ik}. In this case, survey measurements $\bar{b}_{ij}$ and $\tilde{b}_{ij}$ consist of both Gaussian and non-Gaussian noises.

The stochastic CRLB is based on the assumption that $\mathbf{h}$ is a Gaussian random vector with the covariance matrix $\mathbf{C_h}$. Using a similar process as the deterministic CRLB, the stochastic CRLB for Gaussian and non-Gaussian mixed noise can be derived as:

$$CRLB\_s(\boldsymbol{\theta}) = tr\left\{\left((\mathbf{XF})^H\mathbf{B}^{-1}\mathbf{XF} + \mathbf{C_h}^{-1}\right)^{-1}\right\}. \tag{22}$$

The following proposition is provided to give a robust check for the proposed CRLBs.

Proposition 1: The proposed CRLB for the mixed noise will become the CRLB for the Gaussian case when the non-Gaussian noise goes to 0.

Proof: Consider the case when all non-Gaussian noise goes to 0 ($b_{ij}=0,\forall i,j$) i.e. $\bar{b}_{ij}\to 0$, $\tilde{b}_{ij}\to 0$, $\bar{d}_i\to 0$ and $\tilde{d}_i\to 0$, then:

$$\lim_{b_{ij}\to 0} f_{v_i}(v_i)=\frac{1}{\pi\sigma_i^2}\exp\left(-\frac{\bar{n}_i^2}{\sigma_i^2}\right)\exp\left(-\frac{\tilde{n}_i^2}{\sigma_i^2}\right). \tag{23}$$

The limit of $g_i(v_i)$:

$$\lim_{b_{ij}\to 0} g_i(v_i)=\frac{1}{\pi\sigma_i^2}\exp\left(-\frac{\bar{n}_i^2}{\sigma_i^2}\right)\left(-\frac{\bar{n}_i}{\left(\sigma_i^2/2\right)}\right)\exp\left(-\frac{\tilde{n}_i^2}{\sigma_i^2}\right). \tag{24}$$

Substituting (23) and (24) into (18) gives the limit of A_{i1}:

$$\begin{aligned}\lim_{b_{ij}\to 0} A_{i1}&=\int_{-\infty}^{+\infty}\int_{-\infty}^{-\infty}\frac{1}{\pi\sigma_i^2}\exp\left(-\frac{\bar{n}_i^2}{\sigma_i^2}\right)\left(\frac{\bar{n}_i}{\sigma_i^2/2}\right)^2\exp\left(-\frac{\tilde{n}_i^2}{\sigma_i^2}\right)d\bar{n}_i d\tilde{n}_i\\&=\frac{4}{\sigma_i^4}\frac{\sigma_i^2}{2}=\frac{2}{\sigma_i^2}.\end{aligned} \tag{25}$$

Substituting (25) into (21) gives the limit of $\mathbf{B}$:

$$\lim_{b_{ij}\to 0}\mathbf{B}=diag\left\{\begin{bmatrix}\sigma_1^2 & \cdots & \sigma_{N_p}^2\end{bmatrix}\right\} \tag{26}$$

Substituting (26) into (20) and (22) shows that both of the deterministic and stochastic CRLBs in the mixed noise environments will become the CRLBs of the Gaussian case when non-Gaussian noise attains 0.

4. Simulation Results

In the simulation, the CIR is modeled as deterministic unknown process and random Gaussian process for the deterministic and stochastic CRLBs respectively. The covariance matrix of random Gaussian process is:

$$\mathbf{C_h}=0.1\cdot diag\left\{\begin{bmatrix}e^{-0/10} & \cdots & e^{-(L-1)/10}\end{bmatrix}\right\}. \tag{27}$$

(27) is based on multipath fading model with an exponential power delay profile. The same model for $\mathbf{C_h}$ has been used in [12] to describe the random Gaussian channel. In fact, (27) is used to describe the channel environment and will not affect the accuracy of the proposed stochastic CRLB because it is a basic assumption that the stochastic CRLB has the prior $\mathbf{C_h}$.

This OFDM system consists of $N=256$ subcarriers and $N_P=64$ pilot channels. The number of multipath components L is 4.The SNR and signal-to-bias ratio (SBR) are defined as:

$$SNR=10\log_{10}\left(\frac{\sigma_s^2}{\sigma_{back}^2}\right),\quad SBR=10\log_{10}\left(\frac{\sigma_s^2}{\sigma_b^2}\right) \tag{28}$$

where σ_s^2 , σ_{back}^2 and σ_b^2 are the variances of the signals, Gaussian noise and non-Gaussian noise, respectively. For convenience, every pilot channel has the same SNR and SBR and the pilot channels are uniformly distributed among all the channels. The number of samples M is determined using the method in the following subsection. The performance of channel estimation is evaluated through comparing its average CRLB, which is given by:

$$CRLB(\boldsymbol{\theta})/N_p. \tag{29}$$

The proposed CRLB derived in this paper for Gaussian and non-Gaussian mixed noise will compare with the CRLB [12] for Gaussian noise. Researches show that the impulsive nature can be well described by the heavy trail distribution, such as Laplace and Cauchy distributions [20]-[21]. To simulate the non-Gaussian noise, Laplace distribution is selected in the simulation. The PDF of Laplace distribution is:

$$f(x)=\frac{1}{2h}\exp\left(-\frac{|x|}{h}\right) \tag{30}$$

where $x=\bar{\mathbf{b}}$ or $x=\tilde{\mathbf{b}}$.

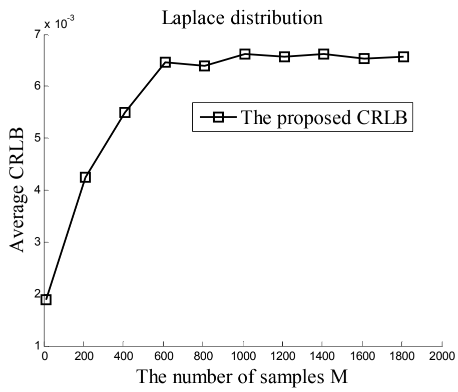

Fig. 1. The determination of the minimum number of samples.

4.1 Case 1: Determining the Minimum Number of Samples

The minimum number of samples M for achieving relatively accurate results using the derived CRLB is a very important issue. It can be seen from [19] that non-parametric kernel method can asymptotically converge to any density function with sufficient samples. This implies that the derived CRLB will converge to its stable value as the number of samples M increases. The minimum M can be determined when the derived CRLBs reach their stable values. Fig. 1 shows the average CRLB versus M for Laplace distribution when SNR=20dB and SBR=-10dB. It can be observed that the proposed CRLB converges when $M\geq 600$. It should be noted that the problem of the determination of the CRLB in the practical environment with non-Gaussian noise will become unsolvable when the survey set of non-Gaussian noise with the minimum M is not available.

4.2 Case 2: Verifying the Proposed CRLBs in Gaussian Noise Environment

This simulation is to compare the proposed CRLB with the Gaussian CRLB [12] in Gaussian noise environment. Since both $\mathbf{n}$ and $\mathbf{b}$ are subject to Gaussian distribution and $\mathbf{v} = \mathbf{b} + \mathbf{n}$, $\mathbf{v}$ is also Gaussian noise. Fig. 2 shows the CRLB comparison with different SNRs. Compared with the CRLB for Gaussian noise, the proposed CRLB can provide almost the same bound in Gaussian noise environment. This means that the proposed CRLB is effective.

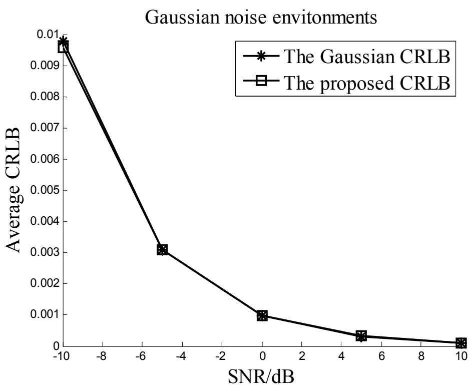

Fig. 2. CRLBs comparison in Gaussian noise environment.

4.3 Case 3: Modeling the PDF Using Non-parametric Kernel Method

This experiment is to evaluate the non-parametric kernel method for estimating the PDF of non-Gaussian noise from survey data. The theoretical and estimated PDFs of the Laplace distributions for $h = 1.58, 0.5, 0.16$ using (30) and (6) are plotted in Fig. 3. Fig. 3 shows that the theoretical and estimated PDFs are basically the same with different h.

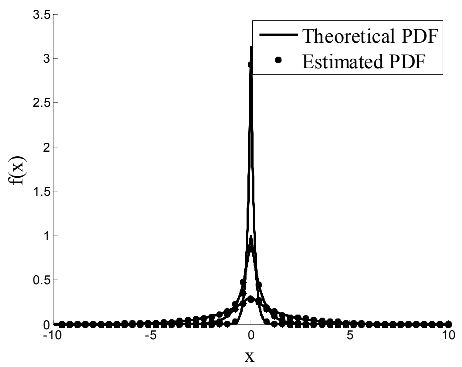

Fig. 3. The PDF comparison for Laplace distribution.

4.4 Case 4: CRLB in Gaussian and Non-Gaussian Mixed Noise Environment

In this case, the performance of the deterministic and stochastic CRLBs ((20) and (22)) in Gaussian and non-Gaussian noise environments are compared. Both $\mathbf{n}$ and $\mathbf{b}$ are added in the channels. The non-Gaussian noise $\mathbf{b}$ is subject to Laplace distribution. Fig. 4 shows the CRLBs versus SBRs with 64 pilot channels and SNR = 20 dB. The results show that the SBRs can reduce the CRLB.

The CRLBs versus the number of the pilot channels with SBR = -10 dB and SNR = 20 dB are shown in Fig. 5. The results show that the stochastic CRLB has the better performance, and the proposed CRLBs will become the Gaussian CRLB when non-Gaussian noise is small, which matches the theoretical analysis in Proposition 1.

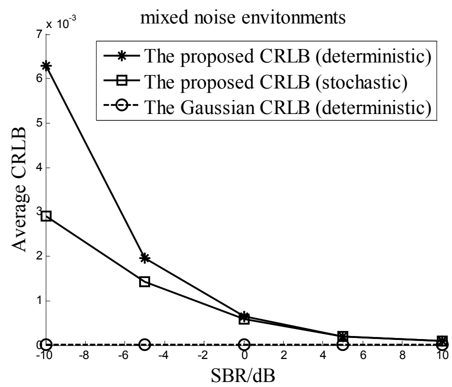

Fig. 4. CRLB versus different SBRs in mixture noise environment.

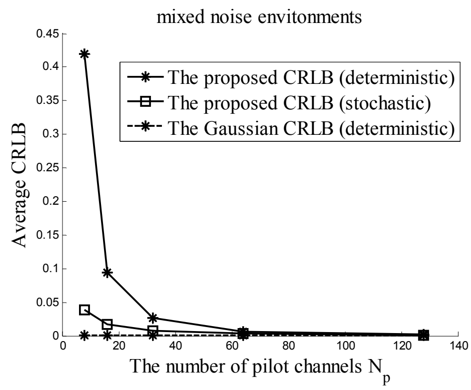

Fig. 5. CRLB versus different N_p in mixed noise environment

5. Conclusions

Determination of CRLB for channel estimation in OFDM system under Gaussian and non-Gaussian mixed noise has been conducted in this paper. The paper first builds the PDF of non-Gaussian noise using the non-parametric kernel method and then derives the CRLBs for channel estimation in Gaussian and non-Gaussian mixed noise environment based on the estimated PDF. The proposed CRLBs consider both the cases that the CIR is a deterministic unknown process or a random Gaussian process. Since the stochastic CRLB has the prior covariance matrix of CIR, the performance of the stochastic CRLB is better than that of the deterministic CRLB as shown in the simulation. This implies that it is useful for a practical system to obtain the CIR information. Since kernel density estimators asymptotically converge to any PDF, the proposed CRLBs based on the non-parametric kernel method are suitable for all noise distributions. This is very important for a practical system because various channel environments may lead to different noise distributions. To provide a deep understanding, the proposed CRLB provides a unified representation for the cases of Gaussian and mixed channel environments. Thus, the unified performance analysis of channel estimation for different channel environments can be achieved by using the proposed CRLB. Moreover, a sanity check is provided to show that the derived CRLB for Gaussian and non-Gaussian mixed noise become the CRLB for Gaussian noise when non-Gaussian noise goes to 0. The paper also points out that there are two important assumptions for the performance analysis of channel estimation in non-Gaussian noise environment: survey measurements of the non-Gaussian noise must be provided and non-Gaussian noise is a stationary process. Without these assumptions, the problem of the determination of CRLB in non-Gaussian environment will become unsolvable since it is impossible to obtain the PDF of the non-Gaussian noise.

Acknowledgements

This work was supported by the Applications and Core Technology University Research of Agilent Technologies (#2831), the Fundamental Research Funds for the Central Universities (ZYGX2009J018), and the National Natural Science Foundation of China (61201275, 61101094 and 61079006). The authors would like to thank the anonymous referees for their suggestions and comments.

References

[1] MORELLI, M., MORETTI, M. Channel estimation in OFDM systems with unknown interference. *IEEE Trans. Wireless Communications*, 2009, vol. 8, no. 10, p. 5338 - 5347.

[2] CAI, J., SHEN, X.M., MARK, J. W. Robust channel estimation for OFDM wireless communication systems - an H∞ approach. *IEEE Trans. Wireless Communications*, 2004, vol. 3, no. 6, p. 2060-2071.

[3] ABDELKEFI, F., DUHAMEL, P., ALBERGE, F. Impulsive noise cancellation in multicarrier transmission. *IEEE Trans. Communications*, 2005, vol. 53, no. 1, p. 94-106.

[4] MENG, J., YIN. W.T., LI, Y.Y., NGUYEN, N.T., HAN, Z. Compressive sensing based high-resolution channel estimation for OFDM system. *IEEE Journal of Selected Topics in Signal Processing*, 2012, vol. 6, no. 1, p. 15-25.

[5] WAN, F., ZHU, W. P., SWAMY, M. N. S. Semiblind sparse channel estimation for MIMO-OFDM systems. *IEEE Transactions on Vehicular Technology*, 2011, vol. 60, no. 6, p. 2569 - 2582.

[6] SHUICHI, OHNO, GIANNAKIS, G. B. Capacity maximizing MMSE-optimal pilots for wireless OFDM over frequency-selective block Rayleigh-fading channels. *IEEE Trans. Information Theory*, 2004, vol. 50, no. 9, p. 2138-2145.

[7] CHOI, J. W., LEE, Y. H. Optimum pilot pattern for channel estimation in OFDM systems. *IEEE Trans. Wireless Communications*, 2005, vol. 4, no. 5, p. 2083- 2088.

[8] BUECHE, D., CORLAY, P., GAZALET, M., COUDOUX, F.X. A method for analyzing the performance of comb-type pilot-aided channel estimation in power line communications. *IEEE Transactions on Consumer Electronics*, 2008, vol. 54, no.3, p. 1074 – 1081.

[9] EDFORS, O., SANDELL, M., VAN DE BEEK, J.J., WILSON, S.K, BÖRJESSON, P.O. Analysis of DFT-based channel estimators for OFDM. *Wireless Personal Communications*, 2000, vol. 12, no.1, p. 55 – 70.

[10] KAY, S. M. *Fundamentals of Statistical Signal Processing: Estimation Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[11] HARRY, V.T. *Detection, Estimation and Modulation Theory: Part I*. John Wiley, 1968.

[12] MORELLI, M., MENGALI, U. A comparison of pilot-aided channel estimation methods for OFDM systems. *IEEE Transactions on Signal Processing*, 2001, vol. 49, no. 12, p. 3065 – 3073.

[13] BERRICHE, L., ABED-MERAIM, K. Stochastic Cramer-Rao bounds for semiblind MIMO channel estimation. In *The Fourth IEEE International Symposium Signal Processing and Information Technology*. Rome(Italy), 2004, p. 119-122

[14] KOZICK, R.J., SADLER, B.M. Maximum-likelihood array processing in non-Gaussian noise with Gaussian mixtures. *IEEE Transactions on Signal Processing*, 2000, vol. 48, no.12, p. 3520 – 3535.

[15] HUANG, J.Y., WAN, Q. CRLB for DOA estimation in Gaussian and non-Gaussian mixed environments. *Wireless Personal Communications,* 2012, Online First™.

[16] STANKOVIC, L., IVANOVIC, V. Further results on the minimum variance time-frequency distribution kernels. *IEEE Transactions on Signal Processing*, 1997, vol. 45, no.6, p. 1650 – 1655.

[17] DJUROVIC, I., LUKIN, V.V. Robust DFT with high breakdown point for complex-valued impulse noise environment. *IEEE Signal Processing Letters*, 2006, vol. 13, no.1, pp. 25 – 28.

[18] SCOTT, D. *Multivariate Density Estimation: Theory, Practice, and Visualization*. Toronto: John Wiley & Sons, 1992.

[19] ELGAMMAL, A., DURAISWAMI, R., HARWOOD, D., DAVIS, L.S. Background and foreground modeling using nonparametric kernel density estimation for visual surveillance. *Proceedings of the IEEE*, 2002, vol. 90, no. 7, p. 1151-1163.

[20] MIDDLETON, D. Man-made noise in urban environments and transportation systems: Models and measurement. *IEEE Trans. Commun.*, 1973, vol. COM-21, no. 11, p. 1232–1241.

[21] ZHANG, Y., WAN, Q., ZHAO, H.P., YANG, W.L. Support vector regression for basis selection in Laplacian noise environment. *IEEE Signal Process. Lett.*, 2007, vol. 14, no. 11, p. 871–874.

About Authors

Jiyan HUANG was born in Jiangxi, China, on 05, Sep, 1981. He received the B.Eng (Hons.), M.Eng., and Ph.D. degrees from the University of Electronic Science and Technology of China (UESTC), Chengdu, China, in 2003, 2005, and 2008, respectively. Currently, he is an Associate Professor at the University of Electronic Science and Technology of China.

Peng WANG received the B.Sc. degree from Xian Jiaotong University, Xian, China, in 1978, and the M.Sc.

and Ph.D. degrees from the University of Saskatchewan, Saskatoon, Canada, in 1995 and 1998, respectively. Currently, he is an Associate Professor at the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore.

Xiaofeng SHEN received the B.Eng degree from Zhejiang Univ., Hangzhou, China, in 1990, and the M.Eng. from the University of Electronic Science and Technology of China, Chengdu, in 1996. Currently, he is an Assoc. Professor at the Univ. of Electronic Science and Technology of China.