# A Memristor Model with Piecewise Window Function

*Juntang YU, Xiaomu MU, Xiangming XI, Shuning WANG*

Dept. of Automation, TNList, Tsinghua University, Qinghuayuan 1, 100084 Beijing, People's Republic of China

yjt11@mails.tsinghua.edu.cn, mxm09@mails.tsinghua.edu.cn,
xxm10@mails.tsinghua.edu.cn, swang@mail.tsinghua.edu.cn

**Abstract.** *In this paper, we present a memristor model with piecewise window function, which is continuously differentiable and consists of three nonlinear pieces. By introducing two parameters, the shape of this window function can be flexibly adjusted to model different types of memristors. Using this model, one can easily obtain an expression of memristance depending on charge, from which the numerical value of memristance can be readily calculated for any given charge, and eliminate the error occurring in the simulation of some existing window function models.*

## Keywords

Memristor, window function, mathematical model.

## 1. Introduction

Memristor is a fundamental two-terminal passive circuit element, which was firstly postulated in 1971 [1]. This circuit element is characterized by a nonlinear relationship between charge $q$ and flux $\varphi$ through it. Its resistance $M$, called memristance, is defined by

$$M = \frac{d\varphi}{dq}. \tag{1}$$

It can be seen that memristance $M$ depends on charge $q$, which is defined as the time integral of the memristor current. Thus, the memristor can be regarded as a nonlinear resistor with memory. Later, the concept of memristor was extended to memristive systems and other circuit elements with memory [2], [3].

A research from Hewlett-Packard labs indicates that the characteristics of a nanoscale thin-film device can be successfully interpreted by the memristor theory [4]. Since then, this promising circuit element has been widely investigated in various areas, ranging from nonlinear oscillators [5], [6] to logic applications [7]. Particular attention is also devoted to modeling the memristor, which is of great significance for designing memristor circuits and analyzing their nonlinear dynamics.

In [4], the nanoscale device is a semiconductor thin-film sandwiched between two metal contacts. More specifically, the film consists of two regions with low and high concentrations of dopants, respectively. Giving the full length $D$ of the film, total resistance $R$ of this device is

$$R = R_{ON}\frac{w(t)}{D} + R_{OFF}\left(1 - \frac{w(t)}{D}\right) \tag{2}$$

where $w(t)$ denotes the length of the high dopant concentration region, and $R_{ON}$ and $R_{OFF}$ are resistances of the film when the dopant concentration of the entire film is on high and low levels, respectively.

Under the assumption of linear dopant drift in a uniform field with average ion mobility $\mu_V$, in [4] the HP physical model is characterized by

$$v(t) = \left(R_{ON}\frac{w(t)}{D} + R_{OFF}\left(1 - \frac{w(t)}{D}\right)\right)i(t), \tag{3a}$$

$$\frac{dw(t)}{dt} = \mu_V \frac{R_{ON}}{D} i(t) \tag{3b}$$

where $v(t)$ denotes the external voltage applied to the device, and $i(t)$ denotes the excited current through the device. Although (3) can yield a linear equation between memristor $M$ and charge $q$, this model doesn't take into consideration boundary nonlinear dopant drift.

To overcome this drawback, a window function $w(D-w)/D^2$ is multiplied to the right side of (3b) [4]. Let state variable $x = w(t)/D$. The HP physical model with window function can be described as

$$v(t) = (R_{ON}x + R_{OFF}(1-x))i(t), \tag{4a}$$

$$\frac{dx}{dt} = \alpha i(t) f(x) \tag{4b}$$

where window function $f(x) = x(1-x)$ and constant coefficient $\alpha = \mu_V R_{ON}/D^2$. Generally, a class of memristor models can be obtained by using different window functions.

According to [4], [8], a sensible window function model should be able to depict linear dopant drift when $w(t) \in (0, D)$, and boundary nonlinear dopant drift when $w(t) \to 0^+$ or $w(t) \to D^-$. However, the model (4) cannot eliminate nonlinear effects when $w(t)$ is around $D/2$. Additionally, it lacks the flexibility in adjusting the shape of the window function $f(x)$ to model different types of memristors. To address these problems, Y. Joglekar and S. Wolf propose another window function $f_p(x) = 1 - (2x-1)^{2p}$ in [9], where $p \in \mathbb{Z}^+$. This window function can satisfy the previous demands well in the case of $p \gg 1$. Besides, different realizations of window function also have been discussed in literature, such as $1-(x-sgn(-i))^{2p}$ in [10] and $1-((x-0.5)^2+0.75)^q$ in [11], where $sgn(\cdot)$ is sign function, $p \in \mathbb{Z}^+$, and $q \in \mathbb{R}^+$.

Replacing $f(x)$ with these window functions in (4b), the explicit functional relation between memristance $M$ and charge $q$ cannot be derived from these models (from a flux-controlled memristor model with window function $f_p(x)$, only a quite complicated relationship between memristance $M$ and flux $\varphi$ can be obtained in the case of $p > 1$ [12]). Thus, in the simulation of these window function models, the numerical methods for solving differential equations have to be applied. There exists the non-negligible simulation error in some cases. This limits the application of these memristor models to design and analyze memristor circuits.

Except window function model, there are many other memristor models [13], [14], [15], which introduce the nonlinear dependence of the state variable derivation and the memristance on the current. These models also suffer from the same problem.

In this paper, a memristor model with piecewise window function is presented to solve this problem. The proposed window function is continuously differentiable and consists of three nonlinear pieces. The single-value function between memristance $M$ and charge $q$ can be obtained from this model, and the numerical value of $M(q)$ can be easily computed for any given $q$ at a considerable precision level. Thus, the simulation error can be eliminated. In addition, the model shows more flexibility than other window function models.

## 2. Memristor Model with Piecewise Window Function

The piecewise window function proposed in this paper is

$$f_{PW}(x) = \begin{cases} (1+(\frac{x-0.5}{a})^{2b})^{-1}, & \text{for } x_0 \le x \le 1-x_0 \\ kx(1-x), & \text{otherwise} \end{cases} \tag{5}$$

where $a \in (0, 0.5)$, $b \in \mathbb{Z}^+$, and $x_0$, $k \in \mathbb{R}^+$. In order to ensure the continuous differentiability of this window function for given $a$ and $b$, $x_0$ and $k$ should satisfy

$$kx_0(1-x_0) - \frac{1}{1+z_0^b} = 0, \tag{6a}$$

$$(1+b)z_0^b - \frac{b}{4a^2}z_0^{b-1} + 1 = 0, \tag{6b}$$

$$z_0 = \left(\frac{x_0-0.5}{a}\right)^2, \tag{6c}$$

$$x_0 < \frac{1}{2}. \tag{6d}$$

There exists the solution of (6) if the given $a$ and $b$ satisfy that

$$\frac{b(1-4a^2)}{4a^2} > 2. \tag{7}$$

This can be shown as follows. Let $h(z) = (1+b)z^b - \frac{b}{4a^2}z^{b-1} + 1$. For any $a$ and $b$ satisfying (7), it can be seen that $h(0) > 0$ and $h(1) < 0$. Hence, there exists a solution $z_0$ of (6b) such that $0 < z_0 < 1$, in view of the continuity of $h(z)$. Then $x_0 = 0.5 - \sqrt{z_0}a$ and $k = 1/(x_0(1-x_0)(1+z_0^b))$ are a pair of solution for (6). In general, we prefer to the smallest possible $x_0$ to enhance the shape controllability of this window function. Thus, the window function $f_{PW}(x)$ is determined by the two parameters $a$ and $b$. It should be remarked that the condition (7) can be easily satisfied by choosing sufficiently large $b$ for any given $a \in (0, 0.5)$.

With proper parameters $a$ and $b$, i.e. $a \to 0.5^-$ and $b \gg 1$, the window function $f_{PW}(x)$ is close to 1 when $0 < x < 1$, which can model the linear dopant drift of memristor. It can also be seen that $f_{PW}(0) = 0$ and $f_{PW}(1) = 0$ from (5), which can model the boundary nonlinear dopant drift of memristor. A family of window functions $f_{PW}(x)$ are illustrated in Fig. 1.

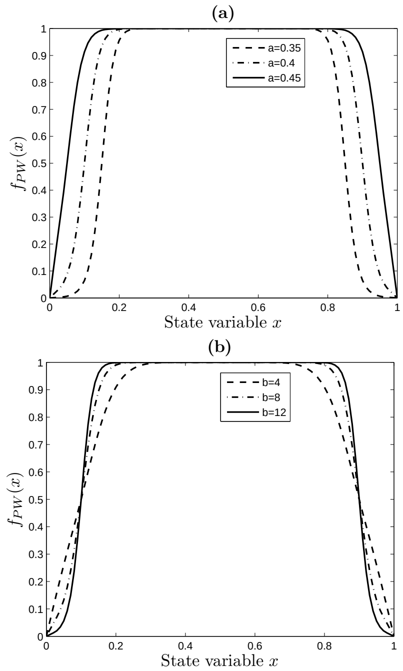


**Fig. 1.** A family of piecewise window functions $f_{PW}(x)$. (a) Given $b = 10$, when $a = 0.35$, 0.4, 0.45, the corresponding values of $x_0$ are 0.0233, 0.0239, 0.0335 and those of $k$ are 0.0909, 1.2771, 10.1102. (b) Given $a = 0.4$, when $b = 4$, 8, 12, the corresponding values of $x_0$ are 0.0806, 0.0306, 0.0198 and those of $k$ are 5.4842, 2.4194, 0.6338. Actually, $x_0$ and $1-x_0$ are smooth connection points of two fragments of (5).

Using this window function, the memristor model can be described by

$$v(t) = (R_{ON}x + R_{OFF}(1-x))i(t), \tag{8a}$$

$$\frac{dx}{dt} = \alpha i(t) f_{PW}(x). \tag{8b}$$

Let $x(0) = x(t)\mid_{t=0}$ and $q(0) = q(t)\mid_{t=0}$. Without loss of generality, assume that $x(0) < x_0$ and $q(0) = 0$, in view of the fact that $q(t)$ can be simply replaced by $q'(t) = q(t) + c$ in other cases where $c$ is a constant. Integrating (8b), it follows that

$$x(t) = 1/(1+\exp(-\alpha k q(t) - c_1)) \tag{9}$$

for $q(t) < q_1$, where $c_1 = \ln x(0) - \ln(1 - x(0))$ and $q_1 = (\ln x_0 - \ln(1-x_0) - c_1)/\alpha k$. Letting auxiliary variable $y(t) = (x(t) - 0.5)/a$ and $g(y) = y + \frac{1}{2b+1} y^{2b+1}$ (see Fig. 2), the differential equation (8b) yields the following equation

$$g(y) = \alpha' q(t) - c_2 \tag{10}$$

for $q_1 \le q(t) \le q_2$, where $\alpha' = \alpha/a$, $c_2 = \alpha' q_1 - g((x_0 - 0.5)/a)$ and $q_2 = (g((0.5 - x_0)/a) + c_2)/\alpha'$. Let $g^{-1}$ denote the inverse function of the one-to-one mapping $g(y)$. It's worth noting that the numerical value of $g^{-1}(u)$ for any $u \in R$ can be readily calculated by Newton Iterative Method in several iterations at a considerable precision level. In particular, the initial value of Newton Iterative Method can be always taken as $y = 0$. And this algorithm converges for any given $u$ in this problem. Then from (10), the variable $y$ has the form of

$$y = g^{-1}(\alpha' q - c_2). \tag{11}$$

It follows that for $q_1 \le q(t) \le q_2$,

$$x(t) = a g^{-1}(\alpha' q(t) - c_2) + 0.5. \tag{12}$$

Similarly, from (8b) it can be obtained that

$$x(t) = 1/(1+\exp(-\alpha k q(t) - c_3)) \tag{13}$$

for $q > q_2$, where $c_3 = \ln(1 - x_0) - \ln x_0 - \alpha k q_2$.

Therefore, combining (9), (12) and (13), the solution of (8b) is

$$x(t) = \begin{cases} (1+\exp(-\alpha k q(t) - c_1))^{-1}, & \text{for } q(t) < q_1, \\ a g^{-1}(\alpha' q(t) - c_2) + 0.5, & \text{for } q_1 \le q(t) \le q_2, \\ (1+\exp(-\alpha k q(t) - c_3))^{-1}, & \text{for } q(t) > q_2. \end{cases} \tag{14}$$

Using (1), (8a) and (14), the memristance $M(q)$ is found to be

$$M(q) = R_{OFF} - (R_{OFF} - R_{ON}) x(t). \tag{15}$$

So far, the solution of (8) has been obtained.

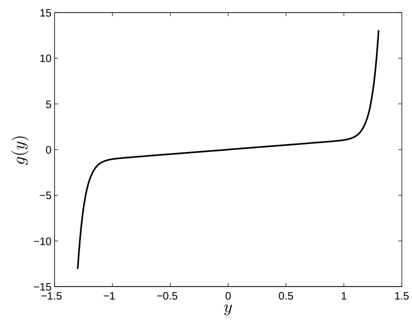


**Fig. 2.** Illustration of the function $g(y) = y + \frac{1}{2b+1} y^{2b+1}$, $b = 10$.

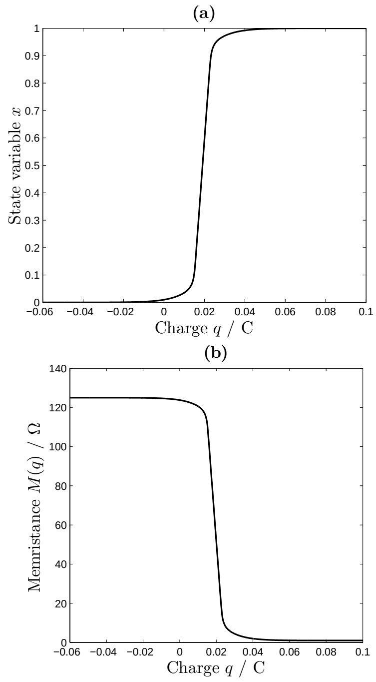


**Fig. 3.** Illustrations of the state variable $x(q)$ and the memristance $M(q)$. $a = 0.4$, $b = 10$, $x(0) = 0.01$, $R_{ON} = 1\ \Omega$, $R_{OFF} = 125\ \Omega$, $D = 10$ nm, $\mu_V = 10^{-10}$ cm$^2$s$^{-1}$V$^{-1}$ (the corresponding values of $x_0$ and $k$ are 0.0239 and 1.2771, respectively).

## 3. Simulation and Analysis

In this section, we do some experiments on the piecewise window function model with different driving sources to test its performances, and indicate main advantages of this model.

When driven by a sinusoidal voltage source, the "pinched hysteresis loop" in the $v - i$ plane can be observed, as shown in Fig. 4. It is consistent with the experimental results in [4]. The $v - i$ characteristic curve "shrinks to a straight line" when the power frequency increases to 10 times larger, as illustrated in Fig. 5. Thus, the simulation results accord with the memristor fingerprint in [16].

There are several benefits of the piecewise window function model. One advantage is that the expression of memristance $M(q)$ can be easily obtained and the numerical value of $M(q)$ can be readily calculated from this expression. Due to the simplicity of the function $g^{-1}$, this memristance expression can be applied as conveniently as the analytic solution to design and analyze memristor circuits. Here we give an example to show the superiority of using the memristance expression.

When the analytic solution of the memristor model described by (4) (of course, $f(x)$ should be replaced by the corresponding window function) cannot be obtained, the

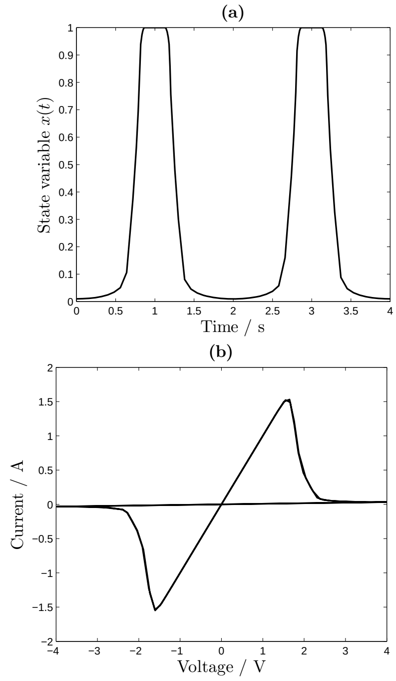


**Fig. 4.** Simulation results of the memristor model with piecewise window function when driving source $V(t) = 4\sin(\pi t)$. The values of parameters are the same as Fig. 3.

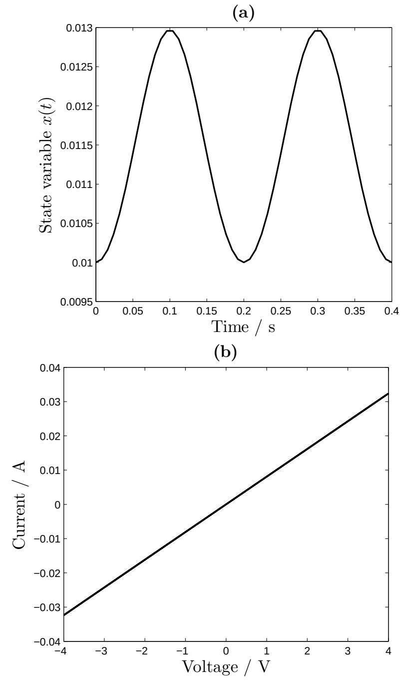


**Fig. 5.** Simulation results of the memriestor model with piecewise window function when driving source $V(t) = 4\sin(10\pi t)$. The values of parameters are the same as Fig. 3.

numerical methods for solving differential equations such as the Runge-Kutta method usually have to be applied in the simulation of the memristor model. This brings non-negligible error when the excitation source has large amplitude or low frequency. Taking the simulation of the memristor model with window function $f_p(x)$ as an example, it can be shown in detail as follows.

Fig. 6 illustrates the simulation results of the memristor model with window function $f_p(x)$ when using the Runge-Kutta method. From (4), it can be obtained that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha \frac{v(t)}{R_{ON}x + R_{OFF}(1-x)} f_p(x). \tag{16}$$

It follows that

$$\int_0^t v(t)\mathrm{d}t = \int_{x(0)}^{x(t)} \frac{R_{ON}x + R_{OFF}(1-x)}{\alpha f_p(x)}\mathrm{d}x. \tag{17}$$

Considering $v(t) = 4\sin(\pi t)$, the above equation (17) yields

$$\frac{4}{\pi}(1-\cos(\pi t)) = \int_{x(0)}^{x(t)} \frac{R_{ON}x + R_{OFF}(1-x)}{\alpha f_p(x)}\mathrm{d}x. \tag{18}$$

Letting $H(x(t)) = \int_{x(0)}^{x(t)} (R_{ON}x + R_{OFF}(1-x))/\alpha f_p(x)\mathrm{d}x$, it can be seen that

$$H(x(t)) = H(x(t+T)) \tag{19}$$

where $T = 2$s. In view of $R_{ON}x + R_{OFF}(1-x) > 0$ and $f_p(x) > 0$ for any $x \in (0,1)$, $H(x)$ is a strictly monotone increasing function of $x$. Therefore, it holds that

$$x(t) = x(t+T). \tag{20}$$

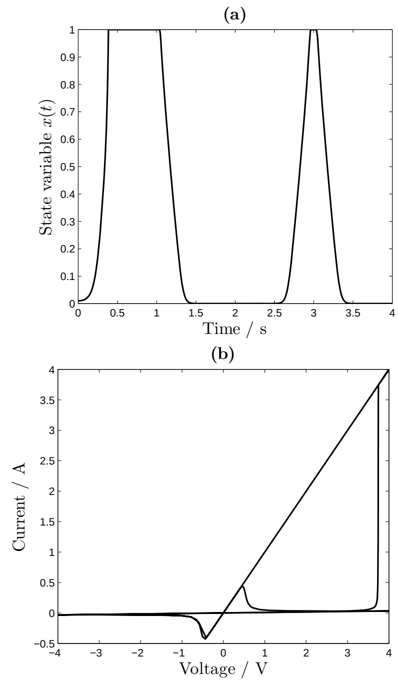


**Fig. 6.** Simulation results of the memristor model with window function $f_p(x)$ when driving source $V(t) = 4\sin(\pi t)$. $f_p = 1-(2x-1)^{2p}$, $p = 3$, the values of other parameters are the same as Fig. 3.

Obviously, the simulation results in Fig. 6 violate the relationship $x(t) = x(t+T)$. This is due to the simulation numerical error. Actually, when $H(x)$ is sufficiently large, $x \to 1$ and $f_p(x) \to 0$. Thus, a small error of $f_p(x)$ can cause a significant difference of $1/f_p(x)$. Therefore, there is a significant difference between the state variable curves of the two excitation cycles in Fig. 6.

When using the piecewise window function model, the numerical value of $M(q)$ can be directly obtained from the memristance expression at a considerable precision level, which avoids numerically solving the differential equations. Hence, the simulation error can be eliminated (see Fig. 4).

A second advantage is that the memristor model with piecewise window function possesses strong flexibility, due to its parameters $a$ and $b$. The linear dopant drift interval of state variable $x(t)$ can be adjusted approximately to $(0.5-a, 0.5+a)$, and the slope of $f_{PW}(x)$ on boundaries is determined by the value of $b$. It can be seen that $f_{PW}(x) \to 1$ for $x \in (0.5-a, 0.5+a)$ and $f_{PW}(x) \to 0$ for $x \in [0, 0.5-a) \cup (0.5+a, 1]$ with sufficiently large $b$ (see Fig. 7). However, the window function $f_p(x)$ with only one controlling parameter $p$ cannot take values close to 0 for a given boundary interval of state variable $x(t)$. Therefore, the shape of the window function $f_{PW}(x)$ can be more flexibly adjusted to model different types of memrisotrs.

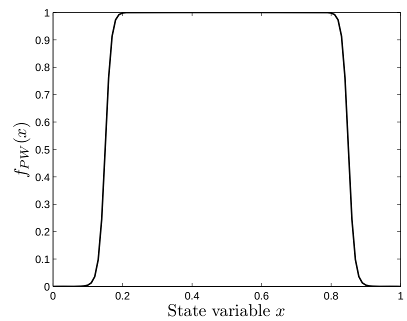


**Fig. 7.** Illustration of the flexible of the piecewise window function $f_{PW}(x)$. $a = 0.35$, $b = 20$ and the corresponding values of $x_0$ and $k$ are 0.0120 and $1.4190 \times 10^{-4}$, respectively.

Introducing the nonlinear dependance of the state variable derivative and the memristance on the current, this model can be developed to characterize memristive systems. In some special cases, the memristance expression depending on charge may also be obtained. For example, when the derivative of the state variable is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha f_{PWL}(i(t)) f_{PW}(x) \tag{21}$$

where $f_{PWL}(\cdot)$ denotes a piecewise linear function, the functional relation between $x(t)$ and $q(t)$ can still be derived. Thus, it can yield the memristance of the memristive systems depending on $q(t)$ and $i(t)$. We will make an in depth study of this issue in future.

# 4. Conclusion

In this paper, a model with piecewise window function is proposed as a proper choice to characterize the memristor. The window function $f_{PW}(x)$ with three nonlinear pieces is exploited to depict linear and boundary nonlinear dopant drifts of memristor. By introducing two parameters $a$ and $b$, the shape of this window function can be flexibly adjusted to model different types of memristors. It is significant that the expression of memristance $M$ depending on charge $q$ can be derived from this model and the numerical value of $M(q)$ can be easily computed for any given $q$. Using the memristor model, the error occurring in the simulation of some existing window function models can be eliminated.

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## About Authors …

**Juntang YU** received the B.S. degree in Control Science and Engineering from Harbin Institute of Technology, Harbin, China in 2011. He is currently pursuing the Ph.D. degree from the Department of Automation, Tsinghua University, Beijing, China. His current research interests lie in the area of modeling and analysis of memristor circuits and systems.

**Xiaomu MU** received the B.S. degree in control science and engineering from Tsinghua University, Beijing, China in 2009. He is currently a Ph.D. candidate in the Department of Automation, Tsinghua University, Beijing, China. His current research areas include nonlinear dynamic system control, piecewise linear identification and memristor device.

**Xiangming XI** received the B.S. degree in Control Science and Engineering from Harbin Institute of Technology, Harbin, China in 2010. He is currently a Ph.D. candidate in the Department of Automation, Tsinghua University, Beijing, China. His research areas includes continuous piecewise linear representation models and identification methods, and global optimization algorithms for CPWL programming.

**Shuning WANG** received the B.S. degree in Electrical Engineering from Hunan University, China in 1982, the M.S. degree and the Ph.D. degree both in the system engineering from Huazhong University of Science and Technology, China in 1984 and 1998, respectively. He was an Associate Professor from 1992 to 1993 and a Full Professor from 1994 to 1995, at the Institute of Systems Engineering, Huazhong University of Science and Technology. He joined Tsinghua University, Beijing, China in 1996. Since then, he has been a Full Professor in Department of Automation, Tsinghua University. He was a Visiting Scholar in the College of Engineering, University of California at Riverside in 1994, and a Visiting Fellow in Department of Electrical Engineering, Yale University from 2001 to 2002. His current research interests are mainly in developing practical methods for nonlinear system identification, control and optimization via piecewise-linear approximation.