MULTICHANNEL SPLINE INTERPOLATION OF DISCRETE SIGNALS

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Abstract

This paper presents the multichannel sampling of a continuous-time signal with division of its frequency band into a few sub-bands. It deals with the combination of this sampling method and spline interpolation with the aim of utilizing their advantageous properties, i.e. low sampling frequency of the multichannel sampling and small error in the spline interpolation series. On the basis of the derived multichannel interpolation series by using of the spline functions, the simulation of two and three channel interpolation of the discrete signals was done.

Keywords:

multichannel sampling and interpolation, analysis filters, synthesis filters, spline interpolation

1. Introduction

Digital processing [8] of signals usually represents a complicated process with a large quantity of data. In connection with this there are high requirments for the transmission rate as well as data memory. Digital coding of signals can reduce these requirements by using suitable methods for sampling and interpolation which are the basic operations in the general coding/decoding model of signals. Effective methods of sampling and interpolation allow us to increase the data compression. An adaptive form of sampling may change the sampling interval with time according to the frequency characteristic of the signal, resulting in a reduction of the average sampling rate. While coding a signal with a high cut-off frequency it is advantageous to carry out multichannel sampling. In this case the sampling frequency in each separate channel is v-times smaller than Nyquist's frequency for the classical sampling, where v is the number of channels. Sampling of the multidimensional signal in general uses various unorthogonal sampling lattices which allow a reduction in the number of samples, etc.

2. Multichannel sampling of continuous signal

Multichannel sampling [1 to 3] is one of the generalizations of classical sampling. It allows to present of a deterministic signal that is bandlimited by using the samples of the separate chnnels with the sampling frequency v-times smaller than Nyquist's frequency f_0 for classical sampling.

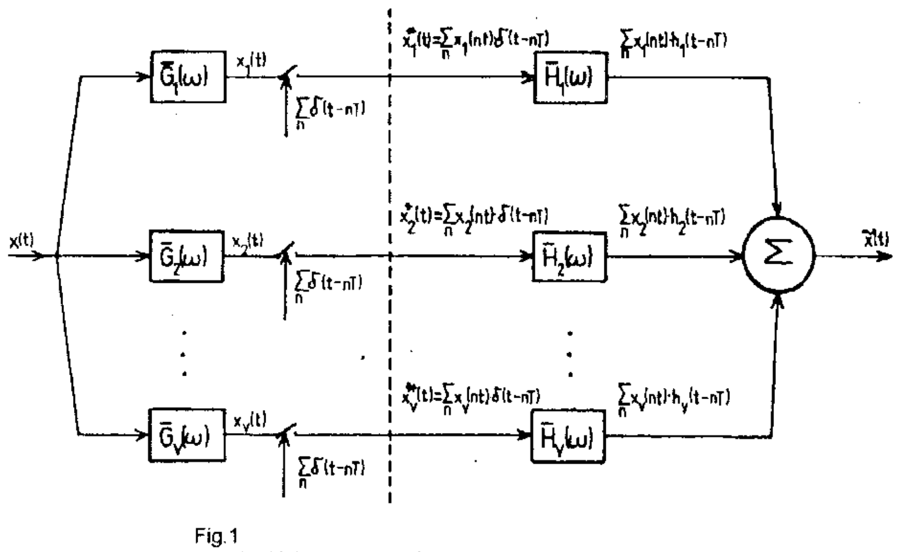

Fig. 1
Multichannel sampling and interpolation.

The block scheme of the multichannel sampling and interpolation is shown in Fig.1. The input continuous signal $x(t)$ is supplied to the analysis filters, while their output signals $x_k(t)$ are sampled according to the sequence of the Dirac impulses $\delta(t-nT)$. The multichannel sampling theorem expresses that the signal $x(t)$, which is bandlimited on the interval $<-\omega_m, \omega_m>$, is exactly determined by the samples $x_k(nT)$ of the output signals $x_k(t)$ of the analysis filters with the common input signal $x(t)$. The sampling period $T=v.\ T_0$ and the sampling frequency in the separate channels is given by

$$f = \frac{1}{v \cdot T_0} = \frac{f_0}{v} \tag{1}$$

where the Nyquist s frequency $f_0 = 2f_m$ with the cut-off frequency of the input signal f_m.

The block scheme of the multichannel sampling and interpolation (Fig.1) contains the analysis and synthesis filters with the frequency responses $\overline{G}_k(\omega)$ and $\overline{H}_k(\omega)$, respectively. An interpolation of the discrete signals $x_k^*(t)$ is caried out in the synthesis filters and their output signals $x_k(t)$ are finally added in the sumator to reconstruct the continuous output signal $x(t)$. If the frequency responses $\overline{G}_k(\omega)$, $k = 1,...,v$ of the analysis filters are not ideal bandlimited functions, then the sampling of the signals $x_k(t)$ on the outputs of the analysis filters with the sampling frequency $f = 1/T = 1/v.T_0$ results in overlapping of the components of the spectrum $\overline{X}_k^*(\omega)$ of the discrete signal $x_k^*(t)$. In general for the perfect reconstructed continuous signal, i.e. $x(t) = x(t)$, we calculate $\overline{H}_k(\omega)$ for the choosen $\overline{G}_k(\omega)$ and the v channels from the following set of equations in a matrix [4],[5] where $\omega_d = 2\pi/T$, while in brackets there valid positive or negative sings for $-\omega_m < \omega \le 0$ and $0 \le \omega < \omega_m$, respectively.

$$\begin{bmatrix} \overline{G}_1(\omega) & \overline{G}_2(\omega) & \dots & \overline{G}_v(\omega) \\ \overline{G}_1(\omega \pm \omega_d) & \overline{G}_2(\omega \pm \omega_d) & \dots & \overline{G}_v(\omega \pm \omega_d) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \overline{G}_1[\omega \pm (v-1)\omega_d] & \overline{G}_2[\omega \pm (v-1)\omega_d] & \dots & \overline{G}_v[\omega \pm (v-1)\omega_d] \end{bmatrix} \begin{bmatrix} \overline{H}_1(\omega) \\ \overline{H}_2(\omega) \\ \dots \\ \dots \\ \overline{H}_v(\omega) \end{bmatrix} = \begin{bmatrix} T \\ 0 \\ \dots \\ \dots \\ 0 \end{bmatrix} \tag{2}$$

A qualitatively diffrent situation,i.e. without overlapping of the components of the spectrum $\overline{X}_k^*(\omega)$, will occur, if the analysis filters are ideal. Consider that the frequency band of the input signal will be divided into v equal sub-bands by using the ideal analysis filters with the zero phase frequency responses. Then perfect reconstructed output continuous signal will be

$$x(t) = \sum_{n=-\infty}^{\infty} x_1(nT)[Si(\pi(ft-n))] + x_2(nT)[2Si(2\pi(ft-n)) -$$
$$-Si(\pi(ft-n))] + x_3(nT)[3Si(3\pi(ft-n)) - 2Si(2\pi(ft-n)) +$$
$$\dots + x_v(nT)[vSi(v\pi(ft-n)) - (v-1)Si(v-1)\pi(ft-n))] \tag{3}$$

The multichannel interpolation according to equation (3) has the following disadvantages:

1. There is an interpolation error caused by its cutting, i.e. by adding the finite number of multichannel interpolation series members, from which the reconstructed continuous signal is calculated.
2. Due to the slow fall down of the interpolation functions, it is necessary to consider, in a given moment, a large number of samples, to get a more precise interpolation.
3. Interpolation functions can only be calculated approximately and they are not very suitable for fast calculations, as they do not use arithmetic operations.

3. System of mirror filters

For the multichannel sampling with dividing of the frequency band of the input signal into sub-bands so far we consider the ideal filters. While utilizing the real filters, which have the fluent crossing of the amplitude response from the passband to the stopband, there will be always overlapping of the components of each spectrum $\overline{X}_k^*(\omega)$.

For the number of channels $v = 2$ we shall propose such a combination of the real analysis and synthesis filters, that the overlapping parts of the spectrum $\overline{X}_k^*(\omega)$ will be supressed and the reconstructed continuous output signal will be delayed when compared to the input signal. Let the frequency response of the low-pass filter

$$\overline{G}_1(\omega) = G_1(\omega)\, e^{j\varphi_1(\omega)} = G_1(\omega)\, e^{-j\omega t_1} \tag{4}$$

where t_1 is time delay of the one, and

$$\varphi_1(\omega_m/2) = -\frac{\omega_m}{2} t_1 \le - \tag{5}$$

is valid. We considere the low-pass filter in the first channel with linear-phase response (Fig.2b) and with amplitude response (Fig.2a), which has falling character in the crossing band with width of $2\Delta\omega$, to be supposed, that $\Delta\omega$ is much smaller than $\omega_m/2$.

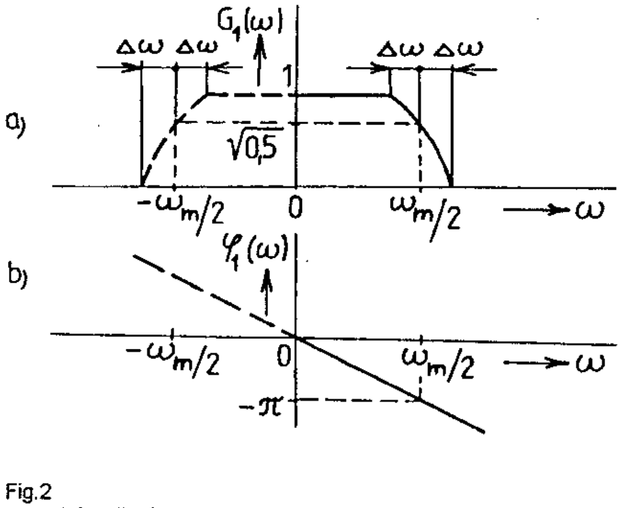

Fig.2
a) Amplitude response $G_1(\omega)$,
b) Phase response $\varphi_1(\omega)$.

Because the frequency band of the input continuous signal is from the upper side limited by cutt-off circle frequency ω_m, in the second channel the analysis filter can be the high-pass one considered as follows

$$\overline{G}_2(\omega) = G_2(\omega)\, e^{j\varphi_2(\omega)} = G_2(\omega)\, e^{-j\omega t_1}\, e^{j\pi/2} \tag{6}$$

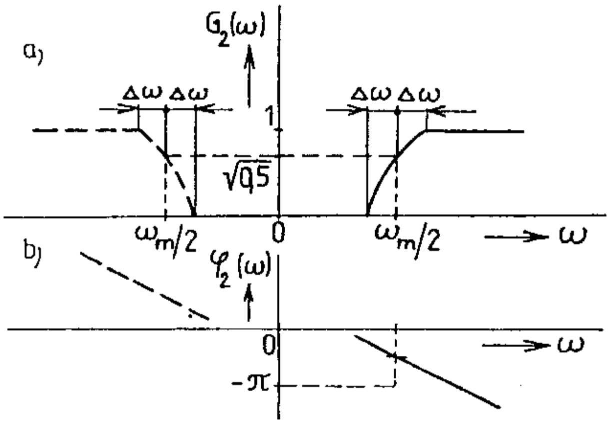

Fig.3
a) Amplitude response $G_2(\omega)$,
b) Phase response $\varphi_2(\omega)$.

The amplitude and phase response are in Fig.3. The amplitude responses are choosen in such a way, that for $\omega \in <-\omega_m, \omega_m$ is valid

$$G_1^2(\omega) + G_2^2(\omega) = 1 \tag{7}$$

This feature is for $\omega > 0$ illustrated in the Fig.4.

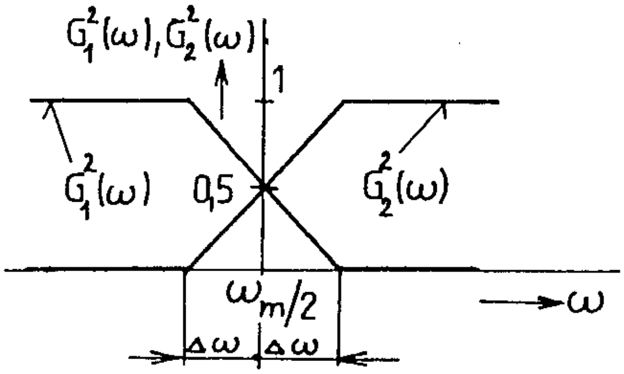

Fig.4
Feature of the mirror filters.

The frequency responses of the synthesis filters can be calculated from the set of the eq.(2), where the right side of the first equation will be $Te^{-j\omega t_c}$. After substitution of the eq.(4),(6) into the matrix of the analysis filters we can write it down for $\omega \in <0, \omega_m$ as follows

$$\underline{B}(\omega) = \begin{bmatrix} G_1\, e^{-j\omega t_1} & G_2(\omega)\, e^{j\omega t_1} e^{j\pi/2} \\ G_1(\omega-\omega_d)\, e^{-j(\omega-\omega_d)t_1} & G_2(\omega-\omega_d)\, e^{-j(\omega-\omega_d)t_1} e^{j\pi/2} \end{bmatrix} \tag{8}$$

The frequency responses of the synthesis filters can be calculated from the equation

$$\begin{bmatrix} \overline{H}_1(\omega) \\ \overline{H}_2(\omega) \end{bmatrix} = \underline{B}^{-1}(\omega) \begin{bmatrix} Te^{-j\omega t_c} \\ 0 \end{bmatrix} \tag{9}$$

from which we are getting

$$\overline{H}_1(\omega) = TG_1(\omega)e^{-j\omega t_1} = T\overline{G}_1(\omega \tag{10}$$

$$\overline{H}_2(\omega) = -jTG_2(\omega)e^{-j\omega t_1} = -T\overline{G}_2(\omega) \tag{11}$$

if we choose $t_c = 2t_1$. For the interval $\omega \in <-\omega_m, 0>$ the validity of eq.(10) and (11) can also be proved. If $\overline{G}_1(\omega)$ and $\overline{G}_2(\omega)$ are given by the eq.(4) and (6), respectively, while eq.(7) is valid for $\omega \in <-\omega_m, \omega_m$, we are talking about so called mirror filters, when the synthesis filters will have the same character as the analysis filters in the separate channels. The resulting system behaves as an ideal delaying network and its response will be

$$\overline{K}(\omega) = \overline{\overline{X}}(\omega)/\overline{X}(\omega) = \frac{1}{T}\sum_{k=1}^{\nu} \overline{G}_k(\omega)\overline{H}_k(\omega \tag{12}$$

i.e. $x(t) = x(t - t_0)$.

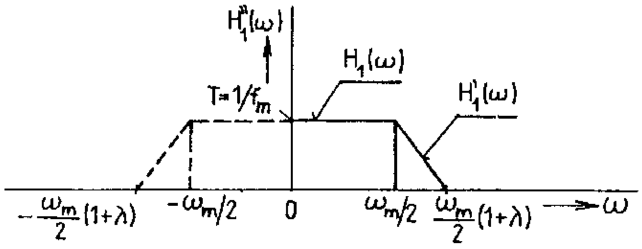

Fig.5
Amplitude response $H_i''(\omega) = H_i(\omega) + H'_i(\omega$.

Further we will show the impulse response of the nonideal low-pass filter with the amplitude response acording to Fig.5 where

$$H_1(\omega) = \{ \begin{matrix} 1/f_m, & |\omega| < \omega_m/2 \\ 0, & |\omega| > \omega_m/2 \end{matrix} \tag{13}$$

$$H_1'(\omega) = \{ \begin{matrix} \frac{\omega_m/2 - |\omega|}{\pi f_m^2 \lambda} + \frac{1}{f_m}, & \omega_m/2 < |\omega| < (1+\lambda)\frac{\omega_m}{2} \\ 0, & otherwise \end{matrix} \tag{14}$$

for $\lambda \in (0,1)$. The resulting impulse response will be

$$h_i''(t) = [(2+\lambda)/2]\, Si[\omega_m t(2+\lambda)/4]\, Si(\omega_m t\lambda/4) \tag{15}$$

From eq.(15) it is apparent, that for values of λ close to zero, the impulse response $h_1''(t)$ will only be slightly different from the impulse response $h_1(t) = Si(\omega_m t/2)$ on the time interval of the main lobe of the function $Si(\omega_m t\lambda/4)$. For $\lambda = 0$

$$h_i''(t) = h_i(t) = Si(\pi f_m t) \tag{16}$$

Analogicaly, we could investigate the impulse response of a nonideal high-pass or band-pass filter. It is possible to come to the similar conclusions as for the nonideal low-pass filter. At the basis of the made analysis we can see that the impulse responses of the mirror filters will be the modified functions $Si(\pi f_m t)$, for the low-pass and band-pass ones.

4. Principle of the multichannel spline interpolation

It can be seen from the eq.(3), that the multichannel interpolation with dividing of the frequency band of the input signal uses the interpolation functions which are linear combinations of the functions $Si(t)$ for separate channels and they can be expressed in the form

$$h_k(t) = kSi(k\pi ft) - (k-1)Si[(k-1)\pi ft] \,, \quad k = 1, \dots, v \tag{17}$$

With the aim of cancelling the above stated disadvantages and to reach a reduction in the number of relevant samples in separate channels, we will make a replacement of these interpolation functions with the optimal spline functions $\phi p_1, \dots, p_s(t)$, which can be expressed as follows [6],[7],

$$\phi p_1, \dots, p_s(t) = s\binom{r+s+1}{s}\sum_{j=0}^{s-1}\binom{s-1}{j}B_{r+j}(t) \tag{18}$$

where

$$r = \begin{cases} 2s \,, & s = 1,2,3,\dots \\ 2s-1 \,, & s = 2,3,4,\dots \end{cases} \quad p_i = r+i-1, \; 1 \le i \le s \tag{19}$$

and $B_m(t)$ are basic spline functions defined by the expression

$$B_m(t) = \frac{1}{\pi}\int_0^\infty Si\left(\frac{u}{2}\right)^m \cos tu \, du =$$

$$= \begin{cases} \frac{1}{(m-1)!}\sum_{n=0}^{[(m/2)-|t|]}(-1)^n\binom{m}{n}\left(\frac{m}{2}-|t|-n\right)^{m-1} \,, & |t| \le m/2 \\ 0 \,, & |t| \ge m/2 \end{cases} \tag{20}$$

where $m \ge 2$ and Gaussian brackets $[(m/2)-|t|]$ denote the largest integer smaller or equal as $((m/2)-|t|)$.

Replacement of the original interpolation functions (eq.(17)) by the optimal spline functions $\phi p_1, \dots, p_s(t)$ can be done in following way

$$h_k(t) = k\phi p_1, \dots, p_s(kft) - (k-1)\phi p_1, \dots, p_s[(k-1)ft] \,,$$

$$k = 1, \dots, v \tag{21}$$

and the generalized series of the multichannel interpolation will have the following form

$$x(t) = \sum_{n=[ft-p_s/2]+1}^{[ft+p_s/2]} x_1(nT)\phi p_1, \dots, p_s(ft-n)$$

$$+x_2(nT)[2\phi p_1, \dots, p_s(2(ft-n)) - \phi p_1, \dots, p_s(ft-n)] + \dots$$

$$+x_v(nT)[v\phi p_1, \dots, p_s(v(ft-n)) -$$

$$-(v-1)\phi p_1, \dots, p_s((v-1)(ft-n)) \tag{22}$$

5. Results of two and three channel spline interpolation of the discrete signals

On the basic of the derived series of the multichannel interpolation by using the spline functions, simulation of two and three channel interpolation of the discrete signals was done. As the input signal $x(t) = (1+t^2)^{-1}$ was used, the practical width of its frequency band is determined by the cut-off frequency $f_m = \pi/2$ Hz.

The dependences of the mean square error ε on the sampling frequency f, when the reconstructed continuous signal $x(t)$ was calculated by eq.(22) on the interval $t \in <-5, 5>$ for the two and three channel interpolation with the spline fuunction $\phi_{3,4}(t)$, are in Fig.6.

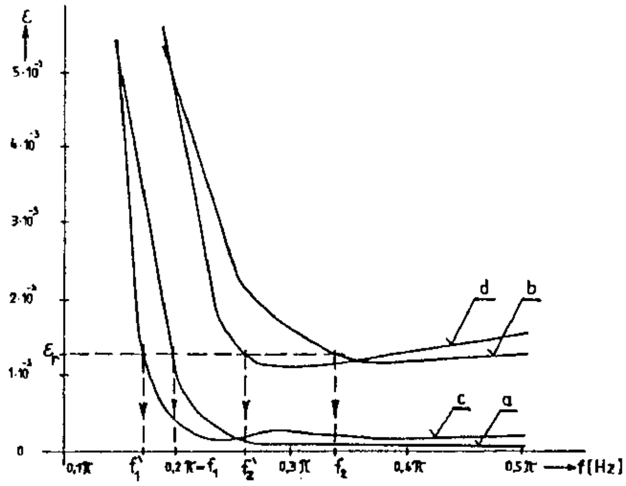

Fig.6
The dependence of the mean square error ε of interpolation on sampling frequency f for two channel interpolation with a) $\varphi_{3,4}(t)$, b) $Si(t)$ and for three channel interpolation with c) $\varphi_{3,4}(t)$, d) $Si(t)$.

Multichannel comparison the same dependences of the mean square error ε which are in Fig.6, too, were calculated for the two and three channel interpolation with the function $Si(t)$. The reconstructed continuous signal $x(t)$ in the given moment was calculated by using the same number of the closest samples in the considered channel, as for the interpolation with $\phi_{3,4}(t)$. From the dependences in Fig.6 we can see the quicker fall down of the mean square error ε for the spline function $\phi_{3,4}(t)$ when comparing it to the fall down the ε for the function $Si(t)$. If we choose as the reference value ε_r of the mean square error, then for the two channel interpolation with the spline function $\phi_{3,4}(t)$ is corresponding sampling frequency $f_1 = 0.2\pi$ Hz and with the function $Si(t)$ will be $f_2 = 0.34\pi$ Hz, and $f_1 < f_2$. Similarly, for the ε_r and for the three channel interpolation with the spline function $\phi_{3,4}(t)$ is the corresponding sampling frequency $f' = 0.17\pi$ Hz and with the function $Si(t)$ will be $f_2' = 0.26\pi$ Hz, and again $f_1' < f_2'$. From abovestated we can see the possibility of a reduction of the sampling frequency for the multichannel interpolation by using the spline functions when comparing with the multichannel interpolation by using of the function $Si(t)$ for the same accuracy of the interpolation. When using the spline functions with bigger support ($p_s > 4$) we can achieve further lowering of the mean square error ε and we would get the analogical dependences as are in Fig.6.

6. Conclusion

Multichannel sampling and interpolation were worked out with the dividing of the frequency band of the input signal into sub-bands, with consideration of the ideal synthesis and analysis filters, i.e. with zero phase response and the amplitude response equal to the constant in the pass-band. If we considered these filters with a linear phase response we should fullfil the requiremen of the coincidence of the resulting linear phase response $\varphi_k(\omega) = \arg[\overline{G}_k(\omega)\overline{H}_k(\omega)$, $k=1,...,v$ in the separate channels. So there will be no error in the reconstructed continuous signal $x(t)$ as the consequence of the different linear phase responses of the synthesis and analysis filters. The reconstructed continuous signal $x(t)$ should, then, only be shifted against the input signal $x(t)$. On the basic of the derived interpolation series for v channels the replacement of the original interpolation functions was made by the optimal spline functions $\phi p_1,...,p_s(t)$. From the results of the made simulation of the two and three channel interpolation of the discrete signal by using the spline functions, there is a possibility of the reduction of the relevant samles in the separate channels, which finally could lead to further decreasing of the sampling frequency. Such a combination of the synthesis and analysis filters (mirror filters) for $v=2$ was proposed, which would be by their features closer to real filters, and they could achieve the perfect reconstructed continuous signal. From analysis of the impulse responses of the real filters with nonideal crossing from the pass-band to the stop-band flows out, that they could also be replaced with a certain error by the spline functions to utilize their advantageous features for multichannel interpolation. Multichannel spline interpolation could be used for example in sub-band coding, or in multichannel digital processing of signals.

7. References

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