Spline Interpolation of Image

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Abstract

This paper presents the use of B spline functions in various digital signal processing applications. The theory of one-dimensional B spline interpolation is briefly reviewed, followed by its extending to two dimensions. After presenting of one and two dimensional spline interpolation, the algorithms of image interpolation and resolution increasing were proposed. Finally, experimental results of computer simulations are presented.

Keywords:

Decimation, interpolation, spline function.

1. Introduction

In digital image processing it is often necessary to solve problems of sampling frequency changing, transformation of a discrete image to analog one, resolution increasing of image pattern and others [1-5].

In these applications, methods of image interpolation can be used from which the spline interpolation is the most widespread [6][7]. In this paper we use of the basic (B) spline functions for image interpolation. The mathematical expression of one-dimensional interpolated signal will be [6]

$$f(t)=\sum_{k=-\infty}^{\infty} c_k \cdot s_k=\sum_{k=-\infty}^{\infty} c_k \cdot s(t-t_k) \tag{1}$$

where c_k are interpolation coefficients determined from the input discrete signal and $s_k(t)$ is the interpolation function. The continuous function $f(t)$ is always given at any time by a finite number of the series members (1). This number depends on the choosen B spline interpolation function, which spans over a finite time interval (support). After presenting of one-dimensional and two-dimensional spline interpolation, design algorithms will be used for image interpolation and resolution increasing of image patterns.

2. One-dimensional spline interpolation

Define B spline functions, which can be used as interpolation functions in (1). Let $S: t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1}$ be a partition of the interval $\langle t_0, t_{n+1}\rangle$ on the real axis. Then B spline function of degree n on S is the following picewise polynomial function [6]

$$B_0(t;t_0,t_1,t_2,\ldots,t_{n+1})=(n+1)(-1)^{n+1}\sum_{k=0}^{n+1}\frac{(t-t_k)^n\sigma(t-t_k)}{\lambda(t_k)}$$

where

$$\lambda(t_k)=\prod_{\substack{j=0\\ j\neq k}}^{n+1}(t_k-t_j),$$

$$\sigma(t-t_k)=\begin{cases}(t-t_k)^0, & \text{for } t>t_k\\ 0 & \text{, for } t\le t_k\end{cases} \tag{2}$$

i.e., it is the unit step function and $n=0,1,2,\ldots$. The cubic spline function $B_3(t)$ is a suitable choise from point of view smoothness of interpolated signal and simple implementation.This piecewise analytic function passes throw the four sampling periods and it is very flexible. Next advantage of the one is good approximation of a given function as well as its first and second derivatives for most engineering interpolation problems, therefore we will consider the B spline function.

For uniformly spaced samples we can express the cubic spline function with its centre at the moment t_k as follows

$$\begin{aligned}s_k(t)&=s(t-t_k)\equiv B_3(t;t_{k-2},t_{k-1},t_k,t_{k+1},t_{k+2})\\&=\tfrac{1}{6T^4}[(t-t_{k-2})^3\sigma(t-t_{k-2})-4(t-t_{k-1})^3\sigma(t-t_{k-1})\\&+6(t-t_k)^3\sigma(t-t_k)-4(t-t_{k+1})^3\sigma(t-t_{k+1})\\&+(t-t_{k+2})^3\sigma(t-t_{k+2})]\end{aligned} \tag{3}$$

where T is the sampling period and it is shown in Fig. 1.

As it is evident from (2) and Fig. 1, cubic spline function has only positive values. This property is very interesting for image processing applications because picture elements (p.e.) should be nonnegative quantities. In case, that p.e. are noisy or have certain amounts of fluctuation, it is often desirable to use positive interpolation functions in order to garantee positive interpolation values of an interpolated image. Moreover, from (3) it is clear, that cubic spline function is shift-invariant. Using it as an interpolation function, the eq.(1) can be considered as a convolution sum, i.e. it is a linear, shift-invariant filtering operation. So the cubic spline function becomes a filter impulse response. Let us consider $\tilde{f}(t)$ at

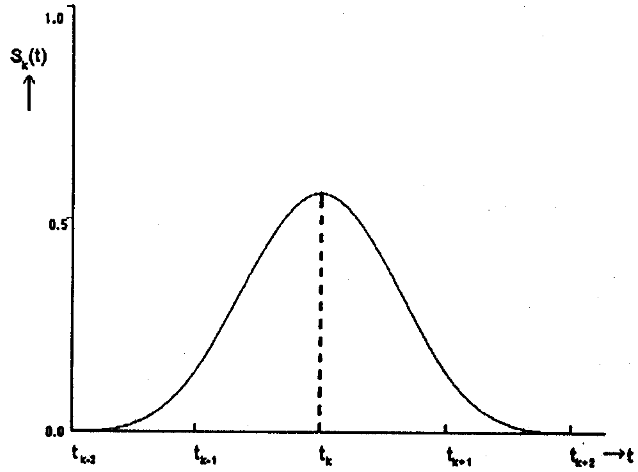

Fig.1
One-dimensional cubic spline function.

$$t = t_k + x \cdot T \text{ and } 0 \le x \le 1 \tag{4}$$

From the definition (3) of cubic spline function and the principle of interpolation by using this one that is shown in Fig.2., the next formula of interpolated signal $f(t)$ on the interval $\langle t_k, t_{k+1}\rangle$ follows out

$$f(t) = \frac{1}{6T^4}\{c_{k-1}\cdot\left[(t-t_{k-3})^3 - 4(t-t_{k-2})^3 + 6(t-t_{k-1})^3\right] +$$

$$+c_k \cdot \left[(t-t_{k-2})^3 - 4(t-t_{k-1})^3 + 6(t-t_k)^3\right] +$$

$$+c_{k+1}\cdot\left[(t-t_{k-1})^3 - 4(t-t_k)^3\right] + c_{k+2}\cdot(t-t_k)^3\} \tag{5}$$

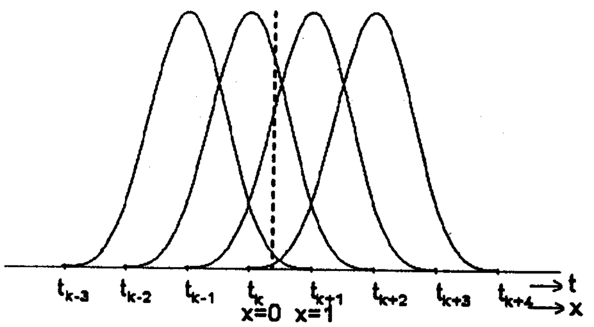

Fig.2
Principle of interpolation by using the one-dimensional cubic spline function.

Substituting (4) into (5) we have

$$f(t_k + x\cdot T) =$$

$$= \frac{1}{6T}\{c_{k-1}\cdot\left[(3+x^3) - 4(2+x)^3 + 6(1+x)^3 - 4x^3\right] +$$

$$+c_k \cdot \left[(2+x^3) - 4(1+x)^3 + 6x^3\right] +$$

$$+c_{k+1}[(1+x^3) - 4x^3] + c_{k+2}\cdot x^3\} =$$

$$= \frac{1}{6T}(b_0\cdot x^3 + b_1\cdot x^2 + b_2 \cdot x + b_3), \quad 0 \le x \le 1 \tag{6}$$

where

$$\mathbf{b} = \mathbf{U}\cdot\mathbf{c_k} \tag{7}$$

$$\mathbf{b} = [b_0, b_1, b_2, b_3]^T, \quad \mathbf{c_k} = [c_{k-1}, c_k, c_{k+1}, c_{k+2}]$$

$$\mathbf{U} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \tag{8}$$

In addition, the value of $f(t)$ in (6) at sampling moment $t = t_k$, i.e. $x = 0$ will be

$$f(t_k) = \frac{1}{6T}(c_{k-1} + c_k + c_{k+1}) \tag{9}$$

The preceeding equation may be written for $k = 1, 2, \ldots, N$ in a matrix form

$$\mathbf{f} = \mathbf{A}\cdot\mathbf{c} \tag{10}$$

where the matrix

$$\mathbf{A} = \frac{1}{6T}\cdot\begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ & & \ddots & & \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \tag{11}$$

has the dimensions $N x N$ and the vectors

$$\mathbf{f} = \left[\tilde{f}(t_1), \tilde{f}(t_2), \cdots, \tilde{f}(t_N)\right]^T, \quad \mathbf{c} = [c_1, c_2, \ldots, c_N]^T$$

It is evident that the matrix **A** is a diagonal, real and symetric one. Each method of interpolation gives the same values of interpolated signal $\tilde{f}(t)$ and original signal $f(t)$ at the sampling moments, i.e. $f(t_k) = f\ (t_k)$

and so

$$\mathbf{f} = \mathbf{A}\cdot\mathbf{c} \tag{12}$$

where $\mathbf{f} = [f(t_1), f(t_2), \ldots, f(t_N)]^T$. Then the interpolation coefficients c_k will be calculated by using the input discrete signal f and the inverse matrix $\mathbf{A}^{-1}$ as follows

$$\mathbf{c} = \mathbf{A}^{-1}\cdot\mathbf{f} \tag{13}$$

3. Two-dimensional spline interpolation

Extending the one-dimensional interpolation formula (1) for two dimensions will be

$$f(h, v) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_{kl}\cdot s_k(h)\cdot s_l(v) =$$

$$=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_{kl}\cdot s(h-h_k)\cdot s(v-v_l) \tag{14}$$

Let us consider an image $f(h,v)$ at two-dimensional interval $(h=h_k+x\cdot T, v=v_l+y\cdot T)$, where $0\le x\le 1$ and $0\le y\le 1$. Analogous to (6) we can writte for two-dimensional interpolated image

$$f(x,y)=\frac{1}{6T}\{\tilde{f}_{l-1}(x)\left[(3+y)^3-4(2+y)^3+6(1+y)^3-4y^3\right]$$

$$+\tilde{f}_l(x)\cdot\left[(2+y)^3-4(1+y)^3+6y^3\right]+$$

$$+\tilde{f}_{l+1}(x)\left[(1+y)^3-4y^3\right]\ +\tilde{f}_{l+2}(x)\cdot y^3\} \tag{15}$$

where

$$f_j(x)=\frac{1}{6T}\sum_{i=0}^{3} b_{ij}\cdot x^{3-i},\quad j=l-1,l+1,l+2, \tag{16}$$

and

$$\mathbf{b_j}=\mathbf{U}\cdot\mathbf{c_j}\ .$$

Next

$$\mathbf{b_j}=[b_{0j},b_{1j},b_{2j},b_{3j}]^T$$

$$\mathbf{c_j}=[c_{k-1,j},c_{kj},c_{k+1,j},c_{k+2,j}]^T$$

and U is given as in (8).

At sampling points (h_k,v_l) , i.e. $x=0$ and $y=0$ we have

$$f(h_k,v_l)=\frac{1}{36T^2}\cdot[(c_{k-1,l-1}+4c_{k,l-1}+c_{k+1,l-1})+$$

$$+4(c_{k-1,l}+4c_{kl}+c_{k+1,l})+$$

$$+(c_{k-1,l+1}+4c_{k,l+1}+c_{k+1,l+1})] \tag{17}$$

for $k,l=1,2,\ldots,N$. The eq.(17) we can again write in a matrix form

$$\mathbf{\tilde{F}}=\mathbf{A}\cdot\mathbf{C}\cdot\mathbf{A} \tag{18}$$

where $\mathbf{\tilde{F}}$ is the matrix of interpolated p.e.. Because, it is valid

$$\mathbf{F}=\mathbf{\tilde{F}}$$

the analogous relation to (12) will be

$$\mathbf{F}=\mathbf{A}\cdot\mathbf{C}\cdot\mathbf{A} \tag{19}$$

where F is a matrix with dimensions NxN made up from p.e. of input image, C - matrix of interpolation coefficients c_{kl}, and A - matrix given by (11). From (19) we can calculate the matrix of interpolation coefficients as follows

$$\mathbf{C}=\mathbf{A}^{-1}\cdot\mathbf{F}\cdot\mathbf{A}^{-1} \tag{20}$$

4. Experimental results

The B spline interpolation method has been applied with real images of raster 256x256 p.e. and these B spline functios were used: zero-$B_0(v)$, first-$B_1(v)$, third-$B_3(v)$ order ones. In Fig.3 for illustration there are graphs of two-dimensional B spline functions $B_1(h,v)$ and $B_3(h,v)$. On the beginning we have carried out image decimation, missing out some p.e. and then we have interpolated an image from retained p.e. Three algorithms have been used for decimation, according to how many p.e.were retained through the horizontal and vertical directions,i.e. decimation 8-each eight, decimation 4-each fourth,decimation 2-each second p.e. We have compared the interpolated image with the original one and evaluated by means of the signal-to-noise ratio

$$SNR=10\cdot\log\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} f_{ij}^2}{\sum_{i=1}^{N}\sum_{j=1}^{N}\left(f_{ij}-\tilde{f}_{ij}\right)^2} \tag{20}$$

where f_{ij} are p.e. of the original image and - p.e. of the interpolated one. We represent the results for two interpolated images LENNA and GIRL in Tab.1 and Tab.2.

The other experiments use the spline interpolation for increasing the image resolution. We have selected from

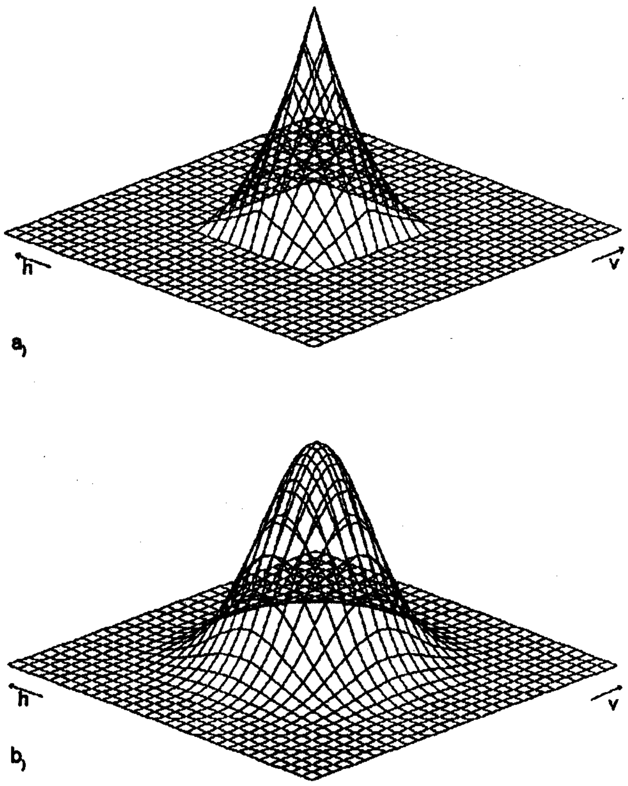

Fig.3
Two-dimensional B spline functions
a)$B_1(h,v)$,
b)$B_3(h,v)$.

B- spline functions	SNR(dB)		
	dec.8	dec.4	dec.2
B_0	13.17	16.66	21.86
B_1	17.95	21.59	26.71
B_3	19.37	23.09	28.05

Tab. 2.
Results for interpolated image LENA

B- spline functions	SNR (dB)		
	dec.8	dec.4	dec.2
B_0	18.75	21.30	25.26
B_1	22.38	25.26	28.46
B_3	24.79	26.28	30.04

Tab. 2.
Results for interpolated image GIRL

the considered images some segments (blocks) of raster 32x32, 64x64, and 128x128 p.e. The obtained segments were enlarged by the spline interpolation to raster 256x256 p.e. The experiments could not be evaluated by an objective criterion because reference images would be needed for comparison. Therefore the results have been evaluated by a subjective criterion from point of view the psychovisual perception of the interpolated images with increased resolution. From results follows, that with increasing support of spline function,the psychovisual perception improves.

5. Conclusion

In the paper the spline image interpolation was presented. The algorithms of image interpolation and resolution increasing were proposed on the basis of B spline functions. From the results presented in *Table 1* and *Table 2* it is seen,that the worst interpolation algorithm is with the B spline function of zero order. The best results were obtained for the B spline function of third order. The zero order interpolation was appeared as the worst on a basis of the psychovisual perception too,when the block structure of interpolated images has affected with an impression of strong interference. Another two interpolation algorithms by means of the B spline function of first and third order have affected with the same impression. From point of view a technical implementation and calculation speed,the most simple interpolation is the zero order interpolation. The complexity of the other algorithms increases with the order of spline function. It would be interesting in the next research to carry out an image interpolation by the optimal spline functions $\phi(t)$ [4] and comparison of achieved results with those of B spline interpolation.

6. References

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