EQUIVARIANT DYNAMICAL SYSTEMS

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Abstract

Our aim is to present some aspects of the mathematical theory of strange behaviour of nonlinear systems, especially of systems with symmetry. Proofs are emitted, the interested reader is advised to references. Our presentation is inevitably selective. We focus on parts of the theory with possible applications to electronic circuits and systems which may display chaotic behaviour.

1. Introduction

It is known that many natural nonlinear systems yield exceedingly complex pseudorandom or chaotic behaviour even though the governing laws are deterministic, and that this behaviour arises primarily from the nonlinear nature of the systems considered.

These surprising phenomena have been already known since Poincaré but until the present their both theoretical and practical significance have not been fully appreciated. To the advances of understanding of these seemingly our intuition contradicting events the advent of the computer was essential, but at the same time it was recognized that the deep mathematical comprehension was equally important. "Reams of computer simulations, without some form of explanation analysis, are not very helpful" [8]. Nowadays we witness the explosive development of the subject. Many profound mathematical results have been uncovered but solutions to some principal questions are still hidden in mystery.

In contrast with linear systems there is not a general theory for nonlinear systems. But there is an agreement that at least some nonlinear systems can be approximately understood by universal model. By creating such paradigmata a mathematical theory of dynamical systems is very useful.

A *system* is something having parts which is perceived as a single entity. The parts making up system may be clearly or vaguely defined. The interesting thing about a system is the way the parts are related to each other. For the systems studied in mathematics, the parts and their relations must be so clearly defined so that we can single out a particular set of these relations as completely characterizing the state of the system. (The mathematical system is evidently oversimplified in comparison with the natural system being modelled.)

A *dynamical* system is one which changes in time (what changes is the state of the system). A mathematical dynamical system then consists of the space of states of the system together with the rule called dynamics for determining the state which corresponds at the given future time to a given present state. The task of mathematical dynamical system theory is then to investigate the patterns of how states change in the long run, after transients died out. We will consider mainly *dissipative* systems (dissipation is the reason why transients cease).

A dynamical system is consequently a pair comprising the state space M (usually R^n) and a collection (f^t) of maps $M \to M$ called the *dynamics* or *flow*. The initial condition $x \in M$ gives after time t a point $f^t x$. When (f^t) is understood we may write $f^t x = x(t)$. The map $t \to x(t)$ is the *trajectory* of $x \in M$. Its image is the *orbit* $O(x)$, and orbit closure is $\hat{O}(x)$. One says that a is a *fixed point* if its orbit consists of a only. For a map f, this means $fa = a$.

The long term behaviour of the trajectory of x is conveniently studied in terms of limit sets. The omega limit set $\omega(x)$ is the subset of M that $x(t)$ approaches as $t \to \infty$. Similarly we can define α-set as subset that $x(t)$ approaches as $t \to -\infty$. A point x (or its trajectory) is *attracted* to the set K if $\hat{O}(x)$ is compact and $\omega(x) \subset$. This implies that the trajectory of x eventually remains in a neighbourhood of K. A set S is attracted to K if every point in S is attracted to K. A point x is *periodic* if there exists $T>0$ such that $x(T) = x$. Then $x(t+T) = x(t)$ for $t>0$. A periodic orbit which is not stationary point is called *cycle*.

Besides asking "what set attracts a given trajectory ?" we can also ask "which trajectories are attracted to a given set ?". An interesting case is that of an attractor. An attractor is a subset Λ of M that has a compact neighbourhood U such that $f(U) < U$ (invariance of U) and $\bigcap_{t \geq 0} f^t(U) =$. It can be shown that, under appropriate conditions, an orbit subjected to small random perturbations will almost certainly approach an attractor [15]. Random fluctuations are always present in physical experiments and numerical iterations of maps by computer. Therefore, the outputs of many physical or computer experiments are attractors.

Some attractors display very remarkable properties, especially some kind of sensitive dependence on initial conditions. If an infinitesimal change $\delta x(0)$ is made to the initial conditions, there will be at time t a corresponding change $\delta x(t)$. We will say that we have sensitive dependence on initial conditions if $\delta x(t)$ grows exponentially with t. This fits well with the commonplace observation that small causes may have large effects. Attractors with this feature are called *strange attractors* [14]. So far there seems to be no universally accepted definition of strange attractor, but this concepts nevertheless useful as such in analysing the results of physical and computer experiments. Time evolutions that show sensitive dependence on initial conditions are called chaotic, and study of chaos in natural phenomena has reached considerable popularity and led to a large number of publications (e.g. [17],[18]).

Strange attractors look strange because they are not smooth curves or surfaces but have noninteger dimension, they are *fractals*. Moreover, while strange attractors have finite dimension, the time frequency analysis reveals a continuum of frequencies.

Poincaré, among many fundamental discoveries in dynamics, found one of the primal sources of chaotic dynamics: the *homoclinic orbits*. He considered trajectories asymptotic, in positive or negative time, to a fixed cycle γ. We call the set of points attracted by γ the *stable manifold* $W^s(\gamma)$; and the stable manifold of γ for the time-reversed flow the *unstable manifold* $W^u(\gamma)$. Now $W^s(\gamma)$ and $W^u(\gamma)$ intersect at all points of γ, but they may also intersect at other points. A point $y \in W^s(\gamma) \cap W^u(\gamma)\backslash$ is called *homoclinic point* belonging to γ. Clearly y is doubly asymptotic to γ, that is $\omega(y) = \alpha(y) = \gamma$. The whole orbit of y (in positive and negative time) consists of homoclinic points. We call a homoclinic point y *transverse* if the stable and unstable intersect transversely at y .

Poincaré's fundamental disclosure was that if y is transverse homoclinic point then there are infinitely many different homoclinic trajectories in $W^s(\gamma) \cap W^u(\gamma$ and they approach γ in very different ways. The significance of this is that in every neighbourhood of the homoclinic point, y contains other homoclinic points exhibiting infinitely many different kinds of limiting behaviour, and the same applies to each of those other homoclinic points, and so on. The system thus exhibits extreme instability with respect to initial values. Poincaré himself was struck by the complexity of such dynamics. His intuition that transverse homoclinic points exist very commonly has proved to be fully justified; in recent years they have been shown to exist in many "natural" dynamical systems.

This kind of behaviour can be demonstrated experimentally in a wide variety of practical nonlinear systems. Completely irregular behaviour of very simple electrical circuits is often cited example and in the present is an object of intensive research. It can destroy the normal function of otherwise carefully designed system and therefore useful theory of nonlinear systems should provide criteria for design of parameter intervals where chaotic behaviour may or can not occur. Detailed "roads to chaos" were investigated; namely scenarios in which a system with simple time evolution becomes chaotic when some parameters are changed.

Let the dynamical system depend on a parameter μ, called a *bifurcation parameter*. Typically, μ will be a real variable, but it could also be a collection of such variables. If the qualitative nature of the dynamical system changes for a value μ_0 of μ ,one says that a bifurcation occurs at μ_0.

The bifurcation theory of attracting sets is particularly interesting. A striking fact is that for some value of the parameter we find only a nonchaotic attracting orbit (a limiting cycle, for example), whereas at some other value of the parameter a chaotic attractor occurs. It is therefore natural to ask how the one comes from the other as system parameter is varied continuously. This problem is complicated, but fortunately, a relatively small number of common transition scenarios are beginning to emerge from theory and experiment. The most spectacular of these scenarios is Feigenbaum's period doubling cascade.

2. Period doubling bifurcation

One of important developments of the theory was the realization that regularities can be expected whenever chaos arises through an infinite series of period doubling bifurcations as some parameter of the system is varied continuously. This "route to chaos" has also been observed in experiments on nonlinear oscillators [9],[10], hybrid optical systems, heat conduction and others [12].

Let f: $M \to M$ be a smooth map. We say that a fixed point a of f is *hyperbolic* if the tangent map $T_a f: T_a M \to T_a$ is hyperbolic, i.e. if the spectrum of $T_a f$ is disjoint from the unit circle $\{z : |z| = 1\}$. (The tangent map for maps of manifolds plays the role of the deprivative *Df* for maps of Banach spaces.) In particular, a is called *attracting* if the spectrum is contained in $\{z : |z| < 1\}$ and a is *repelling* if the spectrum is in $\{z : |z| > 1\}$.

Let us restrict the time to integer values, in terms of some arbitrary unit, then the time evolution is described by the iterates of a single map $f = f^1$. We then have a *discrete time dynamical system*. For such a system let be defined a parametrized family of maps $(f\mu)$ for $\mu \in J$ (where J is an interval of R). Suppose that $\mu_0 \in J$ and that f_0 has a hyperbolic fixed point x_0. Then by implicit function theorem (see e.g. [16]) x_0 is an isolated fixed point of f_0 and there is a differentiable function $\mu \in x(\mu)$ defined in a neigbourhood of μ_0 in J such that $x(\mu)$ is an isolated fixed point of f and $x(\mu_0) = \mu$. In view of this we should look for bifurcations at values of μ such that $x(\mu)$ is not hyperbolic.

Suppose now that the only eigenvalue of $D_{x0}f$ on the unit circle is simple and equal to -1 . Then generically (for a discussion of "genericity" see [16], p. 44) a *period doubling bifurcation* occurs, i.e. a periodic orbit of period 2 is present either for $\mu > \mu_0$ (direct bifurcation) or for $\mu < \mu_0$ (indirect bifurcation).

The period doubling bifurcation is also called *flip* or *subharmonic bifurcation*.

Flip bifurcation for a map may occur in succession and accumulate to a limit, with the attracting fixed point successively being replaced by periodic orbit of period 2^n for $n = 1,2, \ldots \to \infty$. The resulting *period doubling cascade* is known as *Feigenbaum bifurcation*. One great interest of period doubling cascade is that remarkable formula holds:

$$\lim_{n\to\infty} \frac{A_n A_{n+1}}{A_{n+1} A_{n+2}} = \delta = 4.66920\,. \qquad (1)$$

where A_nA_{n+1} are intervals between the places where the successive period doubling occur. These points accumulate to A_∞ . Beyond this point the system behaves chaotically, but there are again "windows" of regular behaviour (see nice pictures in [9], [10]).

This remarkable formula was discovered by M. Feigenbaum, a physicist at Los Alamos numerically. He was playing day and night with computer [17]. Remarkably, δ is *universal* in the sense that many different families ($f\mu$) give the same value δ. Behaviour of systems depend not on the detailed physics or model description but rather on some general properties of the system.

Having discovered the universality of δ experimentally, Feigenbaum went on to propose an explanation of it which was inspired by the renormalisation group approach to critical phenomena in statistical mechanics, but he did not give a complete mathematical treatment of the question. This was later supplied by O. Lanford [12]. Interestingly his mathematically rigorous proof was computer - assisted.

Bifurcation to chaos through period doubling bifurcations appears to be a common feature of systems which are approaching so called *homoclinic tangency* [16] between stable and unstable manifold of period orbit. These manifolds at first do not intersect, become tangent and then intersect transversely. The results of Newhouse [13] then imply that we can expect infinitely many stable periodic orbits to coexist near those at which the tangency occurs.

There is very remarkable fact that complicated nonlinear systems with more then one degree of freedom have the same bifurcation diagram as the seemingly simple one-dimensional unimodal system so called *logistic model*. An intuitive argument for this is the following: Should the specific iteration function contract *N*-dimensional volumes (a dissipative process), then in general is a one direction of slowest contraction, so that after a number of iterations the process is effectively one-dimensional.

Period doubling bifurcations has been observed in many experiments. Perhaps the simplest example is that of periodically dripping tap : as the tap is opened more the period doubles (under certain conditions) and eventually the stream of water becomes chaotic.

The measured values of Feigenbaum's universal number δ are consistent with the theory [12], but they are accurate to only about 5% at best because the rapid convergence rate of the doubling sequence makes it very difficult to observe many doubling.

Behaviour of this kind is often present in nonlinear electronic circuits. It is very likely that it has frequently been observed in laboratory experiments, but that these observations have been misinterpreted with failures. A remarkable exception is the paper of van der Pol and van der Mark that appeared as early as 1927 [9]. They studied a neon bulb oscillator. They observed a creation of subharmonic 1/2, 1/4, e.t.c. with intervals of "irregular noise". They used a telephone receiver as an indicator. We know now that they listened to chaos.

In [18] is reported a direct measurement of a bifurcation diagram for a driven nonlinear semiconductor oscillator, showing successive subharmonic bifurcations to *f/32* , onset to chaos, noise band merging, and intensive noise-free windows. The overall diagram closely resembles that computed for the logistic model. Prof. Chua and his co-workers have been working on application of modern mathematical methods and numerical algorithms on these problems for two decades with a remarkable series of publications. So called *Chua's circuit* is at present object of intensive research. For other examples see the list of references, especially two nice articles by Prof. Hasler [9],[10] and also two special issues [19],[20].

3. Equivariant dynamics

Symmetries change the types of bifurcations that may be expected in dynamical systems and therefore the existence of symmetries should be carefully noted. As we shall see, the simplest nontrivial symmetry $Z_2 = \{\pm 1\}$ acting on R may be expected, for example, to affect period doubling cascades and lead naturally to merging of attractors.

Discrete dynamics with symmetry is determined by a mapping $f\colon M \to M$ which commute with the action of a group G so that

$$f.\tau(x) = \tau.f(x) \qquad \text{for all } \tau \in G, x \in M. \qquad (2)$$

One also says that f is G-equivariant.

If f is iterated, then typically the successive images of a point x settle down towards some attractor, which ranges in complexity form a single point to an intricately structured chaotic set. The symmetry of f places constrains on the form of the attractor. In particular, we can define the symmetry group Σ_A of the set A to be the set of $\tau = \Sigma$ that leaves A invariant, that is, such that $\tau(A) =$. The symmetry of a point x is then defined as the *isotropy subgroup*

$$\Sigma_x = \{\tau \in G : \tau.x = x\}, \qquad \Sigma_x \subset G. \qquad (3)$$

Moreover, let

$$\mathrm{Fix}\,(\Sigma)=\{\, y\in M:\ \sigma y=y,\ \ \forall\sigma\in\Sigma\,\} \quad (4)$$

be the fixed-point space and assume, that G acts absolutely irreducibly on M, that is, that the only liner maps on M that commute with G are scalar multiples of the identity, thus [2], [3]

$$(df)_{0,\lambda}=c(\lambda)\, I\,. \quad (5)$$

Let g: $R^n \times R \to R^n$ be a one-parameter family of G-invariant mappings. We assume that G acts absolutely irreducibly on R^n. It follows that x(0) is a "trivial" fixed point for g and

$$(dg)_{0\lambda}=c(\lambda)\, I\,. \quad (6)$$

For the period doubling bifuracation it must be $c(0)=-1$. The largest subgroup that leaves Fix (Σ) invariant is $N(\Sigma)$, the normalizer of Σ in G. It follows that $g|_{(\mathrm{Fix}(\Sigma)\times}$ commutes with the action of $N(\Sigma)|_\Sigma$. When dim Fix $(\Sigma)=1$, either $N(\Sigma)=\Sigma$ or $N(\Sigma)|_\Sigma \cong Z$ (where $Z_2=\{\pm1\}$ is the simplest nontrivial symmetry) [2].

In the first case, we expect a standard period doubling to occur. In the second case, when $N(\Sigma)|_\Sigma \cong Z$, as the parameter is varied, we might expected the trivial fixed point to undergo a bifurcation to a nontrivial one and each of the nontrivial fixed points then undergoes a period doubling sequence. Each of these sequences seems to behave like the simple logistic equation. This results in a very complicated behaviour with chaotic regimes.

Two more interesting phenomena are worth mentioning with connection with symmetry: *symmetry breaking* [6], where the attractor (usually just a point) loses symmetry, and the reverse process, *symmetry creation* [3].

Suppose that g is G-equivariant and G acts strongly irreducibly on R^n. The symmetry of solution x was defined to be the isotropy subgroup Σ_x. Note that $\Sigma_0=G$, that is, the trivial solution $x=0$ enjoys the full symmetry of the group. Suppose there is a singularity at the origin of g (i.e., $g=\partial g/\partial x=0$, a necessary condition for bifurcation) and the trivial solution is stable for $\lambda<0$ and unstable for $\lambda>0$. Then, as λ is varied through 0, the system jumps to a new state $x\neq0$ and the new solution will have less symmetry then the old. The symmetry has broken spontaneously. (For example, consider the buckling of an Euler column: in the planar model the vertical column enjoys Z_2 symmetry before it buckles and no symmetry after.)

Chaotic attractors can undergo a reverse process, *symmetry creation*. This phenomenon is called a *crisis* and its prevalence in symmetric dynamics is explained by a theorem of Chossat and Golubitsky [3]. Symmetry increasing crises have been observed experimentally by Ashwin [1] in networks of three and four coupled identical electronic oscillators.

4. Conclusion

The aim of the article was to present some basic concepts and ideas of the mathematical dynamical system theory, especially of systems with symmetry. Respect was paid to possible applications to the theory of irregular behaviour of nonlinear electronic systems. The presentation is necessarily incomplete and without mathematical proofs. The reader, who is not supposed to be a mathematician, is advised to the selective list of references. The success of "chaos" has made it a fashionable topic and the number of publications in this field is growing exponentially. It is hoped that this article will provide at least a guideline to it.

References

[1] ASHWIN, P.: Symmetric chaos in systems of three and four forced oscillators, Nonlinearity 3 (1990), pp.603-618.

[2] CHOSSAT, P.- GOLUBITSKY, M.: Iterates of maps with symmetry SIAM J. MATH. ANAL. 19 (1988), pp.1259-1270.

[3] CHOSSAT, P.- GOLUBITSKY, M.: Symmetry increasing bifurcation of chaotic attractors, Physica D32 (1988), pp.423-443.

[4] CHUA, L.O. - LIN, G.N.: Canonical realization of Chua's circuit family, IEEE Trans. CAS 37 (1990), pp.885-902.

[5] FIELD, M.J.: Equivariant dynamics in Multiparameter Bifurcation Theory, Golubitsky M. and Guckenheimer J.M. (Eds.), Contemporary Mathematics AMS, Vol.56, (1986), pp.69-96.

[6] GOLUBITSKY, M. - SCHAEFFER, D.: A discussion of symmetry, and symmetry breaking, in Proc. Symposia Pure Math., Vol.40, Part 1, Orlik P. (Ed.), pp.499-515, Amer. Math.Soc.,1983.

[7] GREBOGI, C. - OTT, E. - YORKE, J.A.: Chaos, strange attractors, and fractal basin boundaries in nonlinear dynamics, SCIENCE 238 (1987), pp.632-638.

[8] GUCKENHEIMER, J. - HOLMES, P.: Nonlinear oscillations, Dynamical Systems and Bifurcation of Vector Fields, Appl. Math.Sci., Vol.42, Springer-Verlag, 1983.

[9] HASLER, M.J.: Electrical circuits with chaotis behaviour, PROC. IEEE 75 (1987), pp.1009-1021.

[10] HASLER, M.J.: Chaos, Proc. ISSSE'92, Paris 1992, pp.241-246.

[11] LANFORD, O.E.: A computer-assisted proof of the Feigenbaum conjectures, Bull.Amer.Math.Soc. 6 (1982), pp.427-434.

[12] LANFORD, O.E.: Period doubling in one and several dimensions, Physica 7D (1983), pp.124-131.

[13] NEWHOUSE, S.E.: Lectures on dynamical systems, in Guckenheimer J., Moser J., Newhouse S.E. (Eds.), Birkhâuser, Boston, 1980.

[14] RUELLE, D.: Strange attractors, The Math.Intelligencer 2, 1980, pp. 126-137.

[15] RUELLE, D.: Small random perturbations of dynamical systems and the definition of attractors, Comm. Math. Phys., 82 1981, pp.137-151.

[16] RUELLE, D.: Differentiable Dynamics and Bifurcation Theory, Academic Press, New York, 1989.

[17] ŠTRUNC, M.: The role of strange attractors in transformation processes of information in the evolution of dissipative dynamical systems (In Czech). Proc. of Conference "SYNERGETIKA '92", Stará Lesná, 1992, pp.54-59.

[18] SHAW, R.: Strange attractors, chaotic behaviour and information flow. Z. Naturforsch. 36a, 1981, pp.80-112.

[19] CHUA,L.O. and HASLER,M. (ed.): Spec. issue on chaos in non-linear electronic circuits. Part A. Tutorials and reviews. IEEE Trans. Circuits Syst. - I. Fundamental ..., Vol. 40, No. 10, pp. 637-786, Oct. 1993.

[20] CHUA,L.O. and HASLER,M. (ed.): Spec. issue on chaos in non-linear electronic circuits. Part A. Bifurcations and chaos. IEEE Trans. Circuits Syst. - I. Fundamental ..., Vol. 40, No. 11, pp. 789-884, Nov. 1993.