Spectral Representation of Discrete-Time Signals and Its Educational Aspects

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Abstract:

Some opinions how to understand and teach spectral representation of discrete-time (DT) signals are discussed in this contribution. There are two excessive ways often used in the educational process. The first one presents discrete - time signal as a product of sampling of continuous - time (CT) signal. The spectral properties of DT signal are then derived from the properties of CT signal and from the peculiarity of the sampling process. Second way builds the spectral description only in the area of DT signals irrespective of the CT signal spectral theory. In our paper, another approach will be presented which prefers the second way but emphasizes the connection and common properties of DT and CT signals. This method enables understanding of new subject - matter because teacher can utilize student's knowledge from the area of CT signals.

Keywords:

continuous-time signal, discrete-time signal, sampling, DFT, equivalent signal.

1. Introduction

There is not sole way how to understand and teach spectral representation of *DT* signals. One of the proved well-known approaches starts of the ideal and then real sampling of *CT* signal. The relation between the spectrum of the original and sampled signals, effect of spectrum periodization, the sampling theorem and other rules of *DT* signal processing are derived from this idea. This approach can be consequently used for "derivation" of definition of *z* transform from Laplace transform and to introduce well-known substitutions $z = exp(sT)$ and $z = exp(j\omega T)$.

The approach mentioned above is based on the theory of *CT* signals and this area is not abandoned because *DT* signal is considered here as a singular case of *CT* signal. However, this fact can later entail trouble for students because in practical applications, the *DT* signal is understood in different sense. In spite of this inconvenience, described method is often used in the opening lectures of various subjects devoted to this problem.

Second widely used approach is based on the building of independent description of *DT* signals without emphasizing of the ambiguous "primarity" of the analogue domain. The elementary *DT* signals are introduced and the so-called discrete-time Fourier set is defined based on the *DT* harmonic signal. This set describes periodical *DT* signal using finite number of spectral components. However, the first inconsistency is often introduced here. Some students can be the worse for this because they will not understand quite correctly the meaning of such practical tools as *DFT* and *FFT*. It may happen if the detailed discussion of physical sense of Fourier coefficients is not performed (e.g. how to obtain magnitudes and initial phases of harmonic components from coefficients, what is so-called degenerated harmonic etc.). The important step how to pass from discrete-time Fourier set to the spectral density of impulse is often omitted, though in case of *CT* signals, this fact is the known part of lectures. Without detailed explanation, the equations for discrete-time Fourier set and its coefficients are directly used for definition of inverse and forward *DFT*.

In some textbooks, the *DFT* is not correctly interpreted and its facilities are often overestimated. For example, we can often find assertion in chapters concerning basic properties of linear *DT* systems that impulse and frequency responses are unambiguously bound by the *DFT*. However, this statement is not true in general because the *DFT* represents only finite number of samples of corresponding signal spectral density and this spectral density is the function of continuous frequency. For this reason, we would prefer *z* transformation over *DFT* in such theoretical cases. Attention would be concentrated on the *DFT* in cases of explanation of numerical implementations of *DT* signal processing.

In our contribution, we will discuss some our experiences obtained during education of *DT* signal processing in the course *Systems, processes and signals (SYPSI)*. Our conception starts from the second aforementioned way but is built consistently on the physical interpretation likewise as in the education of *CT* signals. The *DT* harmonic signal as the basic building block of arbitrary signals plays important role here. Some of its basic properties are distinct from ones of *CT* harmonic signal. These differences are important for understanding of some specific phenomena in the *DT* area.

2. Our conception

The basis of our conception is the thesis that the relation between continuous- and discrete-time signals cannot be understood from the hierarchy point of view. In other words, it is not suitable to understand one of these signal groups as a specific case of the other. The general regularities are true inside each of both groups. The well-known conceptional and mathematical formalism is built for their description. On the other hand, some analogies exist between both groups and it is useful to study them. There are several reasons for it. These analogies enable better understanding of phenomena in the *DT* systems because it can be based on the previous experiences with the regularities inside the analogue systems. Consequently, the lectures devoted to the conversion of continuous- and discrete-time signals can be included to the teaching process.

We start teaching of spectral representation of *DT* signals with the explanation of possible ways of signal generation. The continuous- to discrete-time conversion represents only one of more possibilities. The last mentioned possibility is analyzed (sampling, spectrum after sampling, aliasing, Shannon condition) with the aim to compare it later with the approach based on the "pure discrete-time" basis.

In following parts, we describe some points that we regard as important for the teaching of the spectral representation of *DT* signals.

3. "Analogue" versus "discrete-time" approach

Already during the explanation of the chain of *CT* signal digital processing it is advantageous to attract attention to two different conceptions of *DT* signal:

1 The sequence of impulses where the information of the signal value in a certain discrete time moment is coded either in the height or other impulse parameter. We talk about the **"analogue" approach** (examples: output signal of sampler or *DA* converter).

2 Signal given by the set of numbers and mathematically described by special type of function - by the sequence. We talk about the **"discrete" approach** (examples: information model of signals of digital processor or signals given in item 1).

In essence, signals of first type are *CT* signals. They can exist in the form of time responses of voltages and currents. Also their spectrum has usual physical interpretation (each spectral component means physical existence of harmonic component).

On the other hand, signals of second type represents only certain abstract information models. They are defined by the set of numbers which can describe for instance states of register in the AD converter. Spectrum of these signals has similar character as the spectrum of first-type signals but the physical interpretations are quite different. The analysis of properties of the *DT* harmonic signal as the basic building stone of arbitrary *DT* signals is the key to the understanding of this problem.

4. Discrete-time harmonic signal

It is known that *DT* signal can be expressed either as a function of discrete time (or other independent variable) or as a function of a nondimensional index (this is typical for the orthodox "discrete" approach). For the first study of *DT* harmonic signal, first approach has proved to be better because it enables better comparison of discrete- and continuous- time harmonic signals. This comparison can be significant for the understanding of physical meaning of spectrum of *DT* signal.

DT harmonic signal can be described using formula

$$s(kT) = A\cos(k\omega_0 T + \varphi \qquad (1)$$

or using corresponding graph in Fig.1. There is evident that:

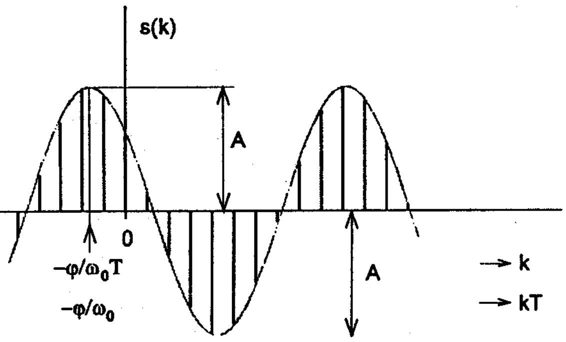

Fig.1 Discrete-time harmonic signal

- *DT* harmonic signal does not to be periodical. This fact is ignored in some textbooks if *DT* harmonic signal is defined as periodic sequence.
- To find out basic signal parameters as magnitude, initial phase and frequency directly from the picture, we must construct "envelope" *CT* signal from samples.
- As shown in Fig.2a) or Eq. (2), there is infinity number of such "envelope" signals:

$$\begin{aligned} s(kT) &= A\cos(k\omega_0 T + \varphi) = A\cos[k(m\omega_s - \omega_0)T - \varphi] \\ &= A\cos[k(m\omega_s + \omega_0)T + \;, \qquad |m| = 0, 1, 2, .., \end{aligned} \qquad (2)$$

where ω_s is the sampling frequency. It means that single *DT* signal could be obtained by sampling of infinity number of *CT* signals. But only one of them fulfills Nyquist's theorem. This signal we call *equivalent signal* to the *DT* signal *s*(*kT*).

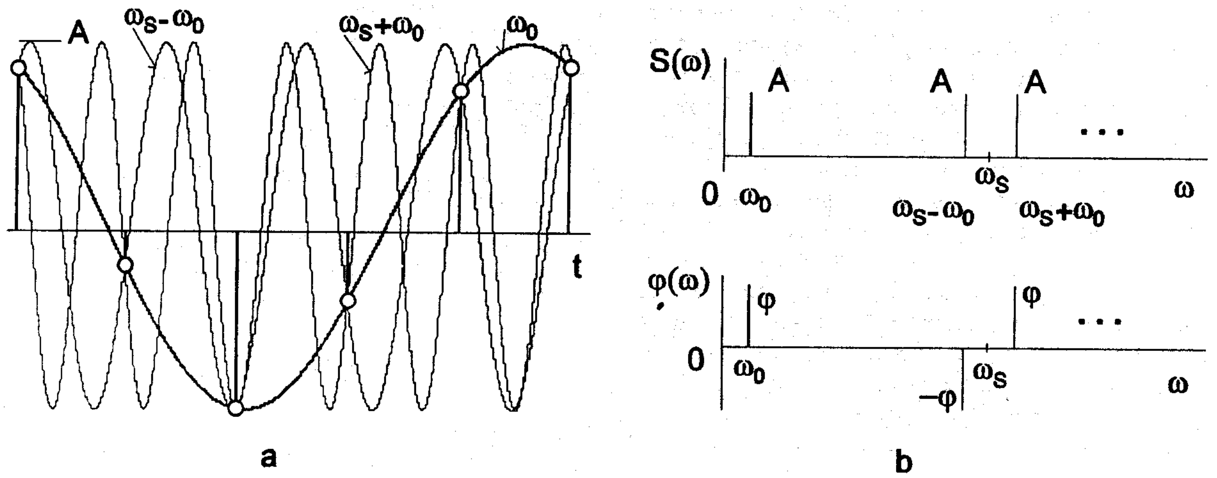

Fig.2 Discrete-time harmonic signal, its "envelope" *CT* signals and corresponding spectral terms

- Aforementioned consideration can lead to the correct understanding of spectral terms in Fig. 2b. Periodically repeated spectral terms only express validity of equation (2), namely that both frequency and initial phase of *DT* harmonic signal are not unambiguous. In distinction from the spectrum of *CT* signal, we obtain physically single but on many frequencies interpreted spectral term.

Special attention must be concentrated on the case if the *DT* harmonic signal is formed by the periodical set. It is suitable to explain how to determine its repeating period (in distinction from *CT* case). It is convenient to pass to the signal expression as a function of nondimensional index and to express period as a number of samples per one period.

5. Fourier set of discrete-time periodical signal

In distinction from *CT* case, arbitrary *DT* periodical signal can be unambiguously compiled from **finite** number of harmonic components. To explain it, there is possible to utilize ambiguous meanings of frequency and initial phase of *DT* harmonic signal. During explanation of Fourier set of *DT* signal, that is the decomposition of periodical signal $s(k)$ with repeating period N to the set

$$s(k) = \frac{1}{N}\sum_{n\in N} \dot{S}(n)e^{jkn-}, \qquad (3)$$

it is convenient to discuss the physical meaning of aforementioned formula which is obvious after following arrangements:

<u>*N* odd:</u>

$$s(k) = \frac{\dot{S}(0)}{N} + \sum_{n=1}^{\frac{N-1}{2}} 2\frac{\left|\dot{S}(n)\right|}{N}\cos\left(kn\frac{2\pi}{N} + \varphi_n\right)$$

$$\dot{S}(n) = \left|\dot{S}(n)\right|e^{j\varphi}$$

<u>*N* even:</u>

$$s(k) = \frac{\dot{S}(0)}{N} + \sum_{n=1}^{\frac{N}{2}-1} 2\frac{\left|\dot{S}(n)\right|}{N}\cos\left(kn\frac{2\pi}{N} + \varphi_n\right) + \frac{\dot{S}(N/2)}{N}(-1$$

It should be noted that *DT* periodical signal with the period N is characterized by N spectral coefficients $\dot{S}(n)$ but it can be compiled only from DC component and $N/2$ (for N even) or $(N-1)/2$ (for N odd) harmonic signals. Second distinction from *CT* case consists in the fact that to obtain magnitudes of corresponding harmonics, we must multiply spectral coefficients by simple constants as seen in aforementioned equations. Third distinction arises for even number of samples when so-called **degenerated harmonic** appears (that is the alternating sequence of type ..1,-1,1,-1,..).

6. DFT

Explanation of differences between the *DFT* and Fourier set of *DT* signal is complicated by the fact that both resulting formulas are identical. Good starting point is the same as in *CT* case: increasing of the period to infinity leads to the model of single impulse and spectral terms converts to the spectral function of continuous frequency. However, the *DFT* arises by the sampling of spectral function and represents only N samples of continuous-frequency spectrum. In this way, the additional effect of periodization of single impulse arises with all known consequences which offer many suggestions for lecturer.

7. Relation between the frequency and time domains

During the analysis of *DT* systems, general relations between their frequency- and time-domain responses are commonly utilized. The time-frequency relations are mostly realized using the direct and inverse *DFT*. This approach is reasonable from the point of view of numerical signal processing and system analysis. However, it is not correct to pick up *DFT* as transformation offering unambiguous relation between the frequency and time domains. We know more general and unambiguous intermediary - the z transformation. Applying the known substitution $z = exp\ (j\omega T)$ to the z - transform of signal yields directly its spectral function as a function of continuous frequency whereas the *DFT* represents only N samples of this function. It depends on the shape of spectral function and on the number of samples if these samples will be representative and if the result of time-frequency conversion will correspond with reality.

We can also refer to the limited facilities of *DFT* by the spectral analysis of periodical *CT* signals and by the computation of the impulse response of the *IIR* systems from the samples of their frequency responses.

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