SOME APPLIED ASPECTS OF RATIONAL HIGHER-ORDER S-Z TRANSFORMATIONS

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Abstract
Some contributions of higher-order s-z transforms are exposed related with the conversion of a continuous-time transfer function into its discrete-time counterpart. The developed algorithm of sensitivity analysis with respect to mapping parameters, prototype coefficients and sampling rate combined with numerical experiments can be efficiently use to motivate the selection of s-z transform and thereby to provide a suitable basis for an optimal solution of a general design problem.

Keywords
s-z transformation, mapping, sensitivity matrix, prewarping

1. Introduction
A derivation of a discrete transform function from its continuous time prototype represents a significant stage in discrete circuit design. In principle, the use of higher-order s-z transforms [1-8] leads to variety of alternative expressions. The choice can be made provided that major properties such as complexity, frequency-frequency linearity, stability, sensitivity etc. [9] are already known. Currently only partial solutions of this problem have been proposed. The method of s.c. sequence of expressions [10] suggests the sensitivity analysis to be implemented at multiple frequency points by a re-evaluation of equations previously generated. Alternatively, the sensitivity functions are developed in terms of a discrete-frequency variable [11]. In both cases the inevitable involvement of the current frequency only complicates the solution.

Here below an attempt is made to consider the impact of the sampling frequency and the basic parameters on the transfer function performance. In result, some stages of the general solution are determined and thereafter combined with numerical experiments to complete the study. The parameter-dependent analysis is developed and sensitivity functions with respect to the sampling rate and sets of mapping parameters and prototype transfer coefficients are found. The prewarping procedure carried out towards the poles and zeros is also included. In effect, all these helpfully assist to implement an extensive comparative analysis and to clarify some major features of discrete system functions obtained via different s-z mappings. The bilinear version is taken throughout the text to serve as a reference and Monte-Carlo simulations are also implemented in parallel to produce alternative assessments.

2. Evaluation procedure of discrete-time transfer functions
Let a discrete circuit has to be designed on the base of a continuous-time transfer function:
\[ H(s) = K_o \left[ \prod_{k=1}^{M} (s - s_{z,k}T) \right]^{-1} \left[ \prod_{k=1}^{N} (s - s_{p,k}T) \right] \]
\[ = K_o \left[ \sum_{k=0}^{M} b_k S^k \right]^{-1} \left[ \sum_{k=0}^{N} a_k S^k \right], \quad M \leq N. \]  
(1)

Here the complex frequency variable \( s = \sigma \pm j\omega \) is weighted by the sampling period \( S = sT \), \( T = 1/f_c \). \( K_o \) is the gain cofactor \( (a_0 = b_0 = 1) \).

The poles and zeros of the prototype (1) are specified as
\[ s_{p,k} = \sigma_{p,k} \pm j\omega_{p,k}, \quad s_{z,k} = \sigma_{z,k} \pm j\omega_{z,k} \]  
(2)

The desired discrete-time transfer function is formulated as
\[ H(z) = K_o \left[ \sum_{k=0}^{KL} B_k z^{-k} \right]^{-1} \left[ \sum_{k=0}^{KL} A_k z^{-k} \right], \]  
(3)

\[ z = \exp(\sigma T \pm j\omega T), \quad K = \max\{M, N\}, \quad L = \max\{m, n\} \]
in the discrete-time domain after processing the prototype (1) by a certain s-z transform
\[ \frac{1}{sT} = F(z) = \left[ \sum_{k=0}^{m} \beta_k z^{-k} \right]^{-1} \left[ \sum_{k=0}^{n} \alpha_k z^{-k} \right]^{-1} \]  
(4)

In this expression the mapping parameters \( \{\alpha_k, \beta_k\} \) have been subject to a prior normalization with respect to \( \max(\|\alpha_k\|, \|\beta_k\|) \). The general expression (4) proves to be effective not only for mappings based on the classical
numerical integration methods produced by a polynomial approximation:

\[ x_{t+1} = \sum_{k=0}^{m} a_k x_{t-k} + h \sum_{k=1}^{m} \beta_k f(x_{t-k}, t_{t-k}), \]  

but also in case of various unconventional methods. In Table 1 a selection of perspective s-z transforms defined on the ground of an equal maximal sampling rate is presented.

Normally, the prototype function (1) is subject to a prior prewarping procedure. Some algorithms suggest the prewarped transfer function \( H^*(s) \) to be obtained preserving at least the shape of the prototype frequency response. Usually, cutoff frequencies or imaginary parts of the poles are involved in the respective re-evaluation [9].

Alternatively, it has been offered [12] to carried out the prewarping procedure over both parts of poles and zeros uniformly denoted \( s_\pm = \sigma \pm j \omega _\pm \). Substituting all powers of \( z = \exp(s_\tau T) \) in (4) by their Euler equivalents the following relation holds:

\[ \alpha ^T \pm j \omega _\tau T = \sum_{k=0}^\infty \sum_{l=0}^\infty \alpha _k \beta _l \exp \left[-(k+l)\sigma _\tau T \right] \cos[(l-k)\omega _\tau T] \pm j \sin[(l-k)\omega _\tau T] \]

\[ \sum_{k=0}^\infty \sum_{l=0}^\infty \alpha _k \beta _l \exp \left[-(k+l)\sigma _\tau T \right] \cos[(l-k)\omega _\tau T] \]

\[ \alpha ^* \pm j \omega _\tau ^* \]

\[ \text{that can be used to evaluate the prewarped poles or zeros } s_\tau ^* = \sigma _\tau ^* \pm j \omega _\tau ^*. \]

In turn, the prewarped transfer function

\[ H^*(s) = K_0 \left[ \prod_{k=1}^M (S - S_\tau ^* k T) \right] \left[ \prod_{k=1}^N (S - S_\tau ^* k T) \right]^{-1} \]

\[ = K_0 \left[ \sum_{k=0}^M b_k^* s^k \right] \left[ \sum_{k=0}^N a_k^* s^k \right]^{-1}, \quad M \leq N \]

is obtained replacing all poles and zeros in expression (1) by their predistorted counterparts:

\[ s_\tau _{p,k} ^* = \sigma _\tau _{p,k} ^* \pm j \omega _\tau _{p,k} ^*, \quad s_\tau _{z,k} ^* = \sigma _\tau _{z,k} ^* \pm j \omega _\tau _{z,k} ^* \]

| Table 1. Some higher-order s-z transforms |  | Table 1. Some higher-order s-z transforms |
|-------------------------------------------|-------------------------------------------|
| Ref. | Mapping function \( F(z) \) | Ref. | Mapping function \( F(z) \) |
| ADAMS MOULTON discrete integration rules | \( 2 + 4z^{-1} \) | GRAHAM-LINDQUIST discrete integrators [1] |
| AM2 | \( 1 + z^{-1} \) | H021 | \( 5 - 4z^{-1} - z^{-2} \) |
| AM3 | \( 5 + 8z^{-1} - z^{-2} \) | H031 | \( 17 - 9z^{-1} - 9z^{-2} + 2z^{-3} \) |
| AM4 | \( 9 + 19z^{-1} - 5z^{-2} + z^{-3} \) | H041 | \( 12 + 48z^{-1} \) |
| AM5 | \( 251 + 64z^{-1} - 264z^{-2} + 106z^{-3} - 19z^{-4} \) | DOSTAL parametric transformations \( a=2.927 \) [3] | \( 1 + az^{-1} \) |
| MILN-SIMPSON discrete integration rules | \( 2z^{-1} \) | (k=0.1) [4] | \( 1 + 16k + 4z^{-1} + (22 - 32k)z^{-2} + 4z^{-3} + (1 + 16k)z^{-4} \) |
| MS2 | \( 1 + 4z^{-1} + z^{-2} \) | TD1 | \( 1 + az^{-1} \) |
| MS3 | \( 3(1 - z^{-2}) \) | TD4 | \( 2.7902(1 - z^{-2}) \) |
| HAMMING discrete integration rules | \( 17 + 51z^{-1} + 3z^{-2} + z^{-3} \) | AL-ALAOUI discrete differentiator [6] |
| HA1/2 | \( 24(2 - z^{-1} - z^{-2}) \) | ALA | \( 1 + 0.5358z^{-1} + 0.0718z^{-2} \) |
| HA2/3 | \( 25 + 91z^{-1} + 43z^{-2} + 9z^{-3} \) | LE BIHAN \( X=0.793 \) discrete integrator [7] |
| HA1/3 | \( 24(3 - 2z^{-1} - z^{-2}) \) | LEB | \( 1 - \frac{X}{2}(1 + \frac{X^2}{2}) \) |
| | \( 24(3 - z^{-1} - 2z^{-2} - z^{-3}) \) | GUROVA-GEORGIEV transformation [8] |
| | \( 1 + 3.8765z^{-1} + z^{-2} \) | NLT | \( 2.9382(1 - z^{-2}) \) |
The new coefficients \( \{a_i^*, b_i^*\} \) are easily computed using the well-known elementary symmetric functions \([13]\):

\[
a_i^* = (-1)^{N-i} \sum_{\sum_{i=0}^{N} \sum_{j=0}^{N} \prod_{r=1}^{i} s_{r,p}^T},
\]

(9)

\[
b_i^* = (-1)^{M-i} \sum_{\sum_{i=0}^{M} \sum_{j=0}^{M} \prod_{r=1}^{i} s_{r,i}^T},
\]

(10)

Next, the s.c. Plug-in-Expansion method \([2]\) is applied to evaluate the required output coefficients \(\{A_k, B_k\}\) in (3):

\[
A_k = \sum_{i=0}^{N} \left[ a_i^* \sum_{j=0}^{k} \left( c_{j,i} d_{n-i-k-1} \right) \right], \quad k \in \{0, NL\},
\]

(11)

\[
B_k = \sum_{i=0}^{M} \left[ b_i^* \sum_{j=0}^{k} \left( c_{j,i} d_{n-i-k-1} \right) \right], \quad k \in \{0, ML\},
\]

(12)

where provisional quantities denoted \(c_{j,i}\) and \(d_{n-i-k-1}\) are evaluated after polynomials in (4) have been raised to integer power of \(i \in \{0, K\}\):

\[
(a_0 + a_1 z^{-1} + \ldots + a_n z^{-n}) = c_{0,0} + c_{1,1} z^{-1} + \ldots + c_{m,n} z^{-n},
\]

(13)

A simple algorithm to compute coefficients \(c_{j,i}\) and \(d_{n-i-k-1}\) is shown in the Appendix A.

### 3. Estimates of frequency response deviations

Some frequency response estimates such as absolute deviation \(\delta(\omega)\) between \(H(S)\) and \(H(z)\) given in (1) and (3)

\[
\delta(\omega) = \left| H(S - j\omega T) - H(z = \exp(j\omega T)) \right|,
\]

(15)

the maximal deviation \(\delta_{\text{max}}(\omega_m)\) and the mean value \(\delta_{\text{IAE}}\)

\[
\delta_{\text{IAE}} = \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \delta(\omega) d\omega,
\]

(16)

are evaluated to illustrate the capacity of the procedure described above.

In Table 2 the results of two numerical design examples are presented. In the first example the bilinear version appears to be superior while in the second one some other mappings indicate superiority. Obviously, the “best” choice is definitely related with the design problem of consideration and looking for an optimal solution a decision have to be made in any particular case.

#### Table 2. Higher-order s-z transforms vs frequency response estimates

<table>
<thead>
<tr>
<th>gain (K_o)</th>
<th>poles</th>
<th>zeros</th>
<th>4-th order 40 Hz band-pass filter with central frequency (f_0 = 1) kHz, (f_c = 8) kHz, (f \in (0,2) kHz)</th>
<th>6-th order 2 kHz band-pass filter with central frequency (f_0 = 2.4) kHz, (f_c = 32) kHz, (f \in (0,8) kHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.078646895</td>
<td>-87.766252 ± j6188.2513, -89.742324 ± j6376.2582</td>
<td>± j5754.6882, ± j6847.0533</td>
<td>-0.048719272</td>
<td>-2854.8133 ± j22177.552, -4999.1447 ± j12849.367, -1076.3239 ± j8145.0575</td>
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<tr>
<td>Numerical example No 1 [11]:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4-th order</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pole</td>
<td>rule</td>
<td>(\delta_{\text{IAE}}, \text{dB})</td>
<td>(\delta_{\text{max}}, \text{dB})</td>
<td>(\omega_{m, s^1})</td>
</tr>
<tr>
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<td>.0005</td>
<td>-.0016</td>
<td>1003</td>
<td>.5811</td>
</tr>
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<td>1006</td>
<td>1.3029</td>
</tr>
<tr>
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<td>.0077</td>
<td>989</td>
<td>1.2729</td>
</tr>
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<td>1011</td>
<td>.2133</td>
</tr>
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<td>.0915</td>
</tr>
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<tr>
<td>NTL</td>
<td>.0018</td>
<td>.0007</td>
<td>945</td>
<td>.0426</td>
</tr>
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</table>
4. Sensitivity estimates with respect to the initial parameters

In fact, the required coefficients \( \{ A_k, B_k \} \) determined by (11) and (12) are functions of (1) the prototype coefficients \( \{ a_j, b_j \} \) and (2) the mapping coefficients \( \{ a_\alpha, \beta \} \). Without loss of generality the next consideration is done in terms of \( \{ s_{p, k}, s_{z, k} \} \) and \( \{ s_{p, \alpha}, s_{z, \alpha} \} \) assuming that \( \{ a_j, b_j \} \) and \( \{ a_\alpha, \beta \} \) can be always found from (9) and (10).

In what follows a procedure to assess the logarithmic sensitivity of coefficients \( \{ A_k, B_k \} \) with respect to the both sets of parameters \( \{ \sigma_r, \omega_r \} \) and \( \{ a_\alpha, \beta \} \) mentioned above is developed. Vectors \( x \) and \( p \) are introduced accounting for the parameters involved:

\[
x = \begin{bmatrix} \sigma_{r, p} ; \omega_{r, p} ; \sigma_{r, z} ; \omega_{r, z} ; \sigma_r ; \beta_r \end{bmatrix} \quad (17)
\]

\[
p = [A : B]^T = \begin{bmatrix} A_0 & A_1 & \cdots & A_{K+1} & B_0 & B_1 & \cdots & B_{K+1} \end{bmatrix}^T
\]

Henceforth, boldface letters indicate vectors and matrices and the superscript \( T \) denotes transpose.

The sensitivity matrix defined as \( S_p = x/p \) is:

\[
S_p = \begin{bmatrix} S_A & S_B \\ S_{\sigma, \omega, \sigma} & S_{\sigma, \omega, \beta} \end{bmatrix} = \begin{bmatrix} S_{A, \sigma, p} & S_{A, \beta, \sigma} \\ S_{B, \sigma, p} & S_{B, \beta, \sigma} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & S_A & S_A \\ 0 & 0 & S_B & S_B & S_B & S_B \end{bmatrix} 
\]

(18)

The block-matrix \( S_{A, A, \sigma, \omega, \sigma, \omega, \sigma} \) is decomposed into sub-matrices with equal dimensions \( 9 \times (K+1) \), their elements express the individual sensitivities with respect to the original poles and zeros. It can be shown that the following matrix factorisations hold:

\[
S_{A, \sigma, \omega} = S_A p_A P_{p_A}, \quad S_{B, \sigma, \omega} = S_B z_A Z_{z_A} \]

(19)

\[
S_{A, \alpha, \beta} = S_A p_A P_{p_A}, \quad S_{B, \alpha, \beta} = S_B z_A Z_{z_A} \]

(20)

Likewise, the block-matrix \( S_{A, B} \) is decomposed into sub-matrices with equal dimensions \( 9 \times (K+1) \), their elements determine the individual sensitivities with respect to the mapping coefficients. Similarly,

\[
S_{A, \sigma, \omega} = S_A p_A P_{p_A}, \quad S_{B, \sigma, \omega} = S_B z_A Z_{z_A} \]

(19)

\[
S_{A, \alpha, \beta} = S_A p_A P_{p_A}, \quad S_{B, \alpha, \beta} = S_B z_A Z_{z_A} \]

(20)

The detailed description of all matrix factors denoted on the right-hand side in (19) and (20) is given in Appendix B.

Looking for a general solution, the well-known matrix norms \( \| \cdot \| \) [13] can be introduced to assess the sensitivity. \( \| S \|_F \) gives a global sensitivity estimate. \( \| S_{A, A, \sigma, \omega, \sigma, \omega, \sigma} \| \) offers separate assessments of the prototype coefficient’s influence. \( \| S_{A, \alpha, \beta} \|_F \) and \( \| S_{A, B} \|_F \) measure the contribution of the mapping parameters. \( \| S_{A, \alpha, \beta} \|_F \) and \( \| S_{A, B} \|_F \) expose the most sensitive output coefficient and the most influential mapping parameter respectively. What is more important, the last two norms assess the precision of the initial coefficient specification required to achieve the prescribed precision of the output coefficients. The most sensitive element \( |S_{ij}|_{max} \) of the matrix \( S \) indicates the highest individual sensitivity.

In Table 3 sensitivity estimates obtained from the second example are presented. Data of bilinear version are given in absolute values and included in the first line but the others are presented in a quotient form to their bilinear counterparts. The data in the last column denoted \( \Delta \) are produced by Monte-Carlo procedure. They measure the area of a tolerance field bounded by the upper and the lower worst-case magnitude values in the frequency range \( f \in (f_{min}, f_{max}) \). During 100 simulations the coefficients \( \{ A_k, B_k \} \) have been computed in terms of parameters \( \{ a_\alpha, \beta \} \) perturbed randomly by up to \( \pm 0.1\% \) and subsequently a frequency response at 500 frequency points has been evaluated.

Comparing the last column with the others one can find out certain correlation between corresponding data. Hence the norms may serve as reliable sensitivity estimates. The contributions specific for each type of mappings and each kind of parameters can be also distinguished.

5. Sensitivity estimates with respect to the sampling rate

The following analysis is done in a classical way in terms of the period \( T \) instead of \( f_c = 1/T \). Implying the coefficients \( \{ a_j^+(s_r), b_j^+(s_z) \} \) are implicit function of \( T \) and using (3) one can define \( S_{T}^H \approx T dH/HdT \) as

\[
S_{T}^H = \sum_{k=0}^{K_L} \sum_{k=0}^{K_L} B_k z^{-k} \quad \sum_{k=0}^{K_L} A_k z^{-k+1}
\]

(21)
Table 3. Higher-order s-z transforms vs sensitivity matrix estimates

<table>
<thead>
<tr>
<th>Rule</th>
<th>$S_{vt}$</th>
<th>$S_{v,vt}^*$</th>
<th>$S_{w,vt}^*$</th>
<th>$S_{v,vt}$</th>
<th>$S_{v,vt}$</th>
<th>$S_{v,vt}$</th>
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<td>AM2</td>
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<td>426.80</td>
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<td>0.01</td>
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<td>1.28</td>
<td>0.35</td>
<td>1.16</td>
<td>1.02</td>
<td></td>
</tr>
<tr>
<td>ALA</td>
<td>8.24</td>
<td>38.43</td>
<td>8.15</td>
<td>20.68</td>
<td>3.92</td>
<td>22.16</td>
<td>2.64</td>
<td></td>
</tr>
<tr>
<td>LEB</td>
<td>104.00</td>
<td>1.44</td>
<td>104.10</td>
<td>212.60</td>
<td>40.59</td>
<td>28.30</td>
<td>4.43</td>
<td></td>
</tr>
<tr>
<td>NTL</td>
<td>0.34</td>
<td>2.61</td>
<td>0.33</td>
<td>0.66</td>
<td>0.25</td>
<td>0.63</td>
<td>1.01</td>
<td></td>
</tr>
</tbody>
</table>

Next, two vectors are introduced denoted $z, \bar{z} \in \mathbb{R}^{(KL+1)}$:

$$z = \begin{bmatrix} z_0^* \ z_{-1}^* \ z_{-2}^* \ \cdots \ z_{-KL}^* \end{bmatrix}^T,$$

$$\bar{z} = \frac{dz}{dt} = \begin{bmatrix} dz_0^* \ dz_{-1}^* \ dz_{-2}^* \ \cdots \ dz_{-KL}^* \end{bmatrix}^T.$$  \hspace{1cm} (22)

and two singular row-matrices $A, B \in \mathbb{R}^{(KL+1) \times (KL+1)}$ are constituted:

$$A = \begin{bmatrix} A \ A_1 \cdots \ A_{KL} \end{bmatrix}, \quad B = \begin{bmatrix} B \ B_1 \cdots \ B_{KL} \end{bmatrix}$$  \hspace{1cm} (23)

Likewise, matrices denoted $\tilde{A}, \tilde{B}$ and $\tilde{B}, \tilde{B}$ are formulated:

$$\tilde{A} = \begin{bmatrix} dA_k/dT \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_k \end{bmatrix}, \quad k \in (0, KL).$$  \hspace{1cm} (24)

$$\tilde{B} = \begin{bmatrix} dB_k/dT \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_k \end{bmatrix}, \quad k \in (0, KL).$$  \hspace{1cm} (25)

Derivatives denoted above are determined in a product form using

$$\frac{dA_k^*}{dT} = \sum_{k=0}^{KL} \frac{dA_k}{dT} \begin{bmatrix} \sigma_{k,p}^* T \pm j\omega_{k,p} T \end{bmatrix},$$

$$\frac{dB_k^*}{dT} = \sum_{k=0}^{KL} \frac{dB_k}{dT} \begin{bmatrix} \sigma_{k,p}^* T \pm j\omega_{k,p} T \end{bmatrix},$$  \hspace{1cm} (26)

in the products $\frac{\partial A_k}{\partial t}, \frac{\partial B_k}{\partial t}, \frac{\partial A_k^*}{\partial t}, \frac{\partial B_k^*}{\partial t}$.

The first cofactors in (26) are found from (9) and (10), but expression (6) is used to derive the second cofactors.

It can be shown that the next relations hold:

$$B \tilde{z} = -\dot{\bar{z}}, \quad A \tilde{z} = -\dot{\bar{z}} \tilde{A} z.$$  \hspace{1cm} (27)

Finally, substituting (22)-(25) and (27) in (21) and assuming $s = j\omega$ one can derive the result given below:

$$S_H^T = S_{Re} + jS_{Im} = T \begin{bmatrix} \tilde{B}z - \tilde{B}z^* \tilde{A}z - \tilde{A}z^* \end{bmatrix} = T \begin{bmatrix} \frac{\tilde{B}z - \tilde{A}z}{\tilde{B}z - \tilde{A}z^*} \end{bmatrix} = T \begin{bmatrix} \tilde{B}z - \tilde{A}z \end{bmatrix}.$$  \hspace{1cm} (28)

In fact, the matrix ratios given above represent real numbers but the coefficient $S_H^T$ can not be evaluated straightforward. The reason is that matrices $A$ and $B$ are noninvertible because of matrix singularity. However, $S_H^T$ could be assessed by means of ratios $N_A/M_A$ valid for an arbitrary vector $z$ ([13], ch.6):

$$\frac{\tilde{A}z}{\tilde{A}z^*/\tilde{A}z^*} \leq \frac{N_A}{M_A}.$$  \hspace{1cm} (29)

where for $z \neq 0$ the following expressions are valid:
N = \sup\left|z_{d}\right|_{F} \geq \left|z_{i}\right|_{F}, M = \inf\left|z_{d}\right|_{F} = \frac{1}{\left|z_{i}\right|_{F}} \rightarrow 0 (30)

respecting the singularity of both matrices in (23). Setting
up M = M = \varepsilon \rightarrow 0, one can write

\left|S_{R}\right| = \varepsilon T \left|x_{E}\right| - \left|z_{i}\right|_{F}, \left|S_{I}\right| = \varepsilon \omega_{\Sigma_{1}} = \frac{T}{\varepsilon} \left|x_{E}\right| - \left|z_{i}\right|_{F}

(31)

However, the factor \varepsilon will disappear taking estimates in a
quotient form as mentioned above.

A couple of parameters are introduced to complete the
analysis evaluated for \omega \in (\omega_{\text{min}}, \omega_{\text{max}}):

\Gamma_{1} = \text{arctg}(\Sigma_{R} / \Sigma_{I}), \quad \Gamma_{2} = \sqrt{\Sigma_{R}^{2} + (\omega \Sigma_{I})^{2}} (32)

Unlike all other estimates considered to this point \Gamma_{2}
is the only frequency dependent one. Alternatively, the
worst case value \Gamma_{2}(\omega = \omega_{\text{max}}) can be used.

In Table 4 sensitivity estimates \Sigma_{R} and \Sigma_{I} valid for
the second example are presented. Data are arranged as in
Table 3. Monte-Carlo sensitivity analysis is implemented as
described above towards \(H(f) = \exp(j\omega T)\). A couple of
expressions are defined as

\Delta_{1} = \text{arctg}(\left|H_{im} / \omega H_{Re}\right|), \quad \Delta_{2} = \left(\sqrt{H_{Re}^{2} + H_{Im}^{2}}\right) (33)

They are evaluated for \omega \in (\omega_{\text{min}}, \omega_{\text{max}}) to measure
the area of tolerance fields. The pseudo-random perturbations
within the range \pm 0.001f_{c} are generated to
determine the sampling rate impact on the output
coefficients \{A_{4}, B_{1}\}. Comparing data in columns 3 and 4
with those in columns 5 and 6 respectively one can
establish a certain correlation between data. Hence, they
are used to assess the sensitivity in this case and the
contributions specific for each mapping can be also
distinguished.

6. Conclusions

A compact procedure to derive a discrete-time transfer
function from its continuous-time prototype employing
rational higher-order s-z transform functions is presented.
An algorithm based on some matrix norms is proposed to
assess the output coefficient sensitivity with respect to the
prototype coefficients, the mapping parameters and the
sampling rate. All operations are completely formalised.
The estimate values appears to be specific for each s-z
transform and represent a reliable basis to discuss the
mapping applied properties. The developed analysis
combined with numerical experiments can be efficiently
used to motivate the selection of s-z transformation and
thereby to provide a suitable base for an optimal design
solution.

| Table 4. Higher-order s-z transforms vs sensitivity estimates with respect to the sampling rate |
|------------------------------------------|-----------------|-----------------|-----------------|-----------------|
| Numerical example No 2 [12]: 6-th order 2 kHz band-pass filter with central frequency \(f_{0} = 2.4\ kHz, f_{c} = 32\ kHz, \omega T \in (0, \pi/2)\) |
| \(\Sigma_{R}\) | \(\Sigma_{I}\) | \(\Gamma_{1}\) | \(\Gamma_{2}(\omega = \pi/2)\) | \(\Delta_{1}\) | \(\Delta_{2}\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| absolute values |
| AM2 | 8.1661 | 7.2746 | 5.1589 | 10.936 | 38.800 | 118.0 |
| relative values |
| AM3 | .84 | .99 | 1.14 | .91 | .99 | .82 |
| AM4 | .59 | 1.00 | 1.47 | .80 | .95 | 1.24 |
| AM5 | .16 | 1.01 | 2.51 | .68 | .91 | 1.83 |
| MS2 | .40 | 3.00 | 2.59 | 2.02 | 1.96 | 2.42 |
| MS3 | .57 | 2.28 | 2.23 | 1.57 | 2.16 | 1.32 |
| HA1/2 | .51 | 1.22 | 1.79 | .90 | 1.50 | 1.29 |
| HA2/3 | .63 | 2.08 | 2.09 | 1.46 | 2.55 | 1.41 |
| HA1/3 | .65 | 1.39 | 1.70 | 1.04 | 1.95 | 1.32 |
| H021 | .61 | .60 | .99 | .60 | 1.74 | .90 |
| H031 | .60 | .56 | 1.78 | 1.04 | 1.51 | 1.28 |
| H041 | .96 | 1.41 | 1.98 | 1.99 | 1.94 | 1.38 |
| TD1 | .79 | .99 | 1.20 | .88 | 1.34 | .92 |
| TD4 | .81 | 4.72 | 2.47 | 3.20 | 3.94 | 4.88 |
| TIK | .56 | 2.25 | 2.23 | 1.55 | 2.17 | 1.27 |
| ALA | .29 | 2.01 | 2.55 | 1.36 | 2.25 | .97 |
| LEB | 278.23 | 23.44 | .09 | 208.35 | 1.88 | 5.93 |
| NTL | .57 | 2.27 | 2.23 | 1.57 | 2.16 | 1.31 |
Appendix A

Suppose coefficients $c_i$, $d_{n-i,k-1}$ in (11)-(14) are elements of two sets of vectors denoted $c_i$, $i \in \{1, NL+1\}$ and $d_i$, $i \in \{1, ML+1\}$. Thus, two quasi-triangular matrices are defined:

$$
C = \begin{bmatrix}
c_0^T & 0 \\
c_1^T & 0 \\
\vdots & \vdots \\
c_{NL+1}^T & 0 \\
\end{bmatrix} \in \mathbb{R}^{(KL+1) \times (KL+1)},
D = \begin{bmatrix}
d_0^T & 0 \\
d_1^T & 0 \\
\vdots & \vdots \\
d_{KL+1}^T & 0 \\
\end{bmatrix} \in \mathbb{R}^{(KL+1) \times (KL+1)}
$$

(A1)

Obviously, their first two lines look as follows:

$$
c_0^T = [1 \ 0 \ 0 \ \ldots \ 0],\quad d_0^T = [1 \ 0 \ 0 \ \ldots \ 0],
$$

$$
c_1^T = [a_0 \ a_1 \ a_2 \ \ldots \ 0],\quad d_1^T = [a_0 \ \beta_0 \ \beta_1 \ \ldots \ 0].
$$

(A2)

A simple procedure can be offered to find the others:

1. Define a circulant matrix $P$ [13] in a form

$$
P = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
\end{bmatrix} \in \mathbb{R}^{K \times (K+1)}, K = \max(M, N) \quad (A3)
$$

2. Compute a couple of auxiliary band matrices $Q_c \in \mathbb{R}^{K \times K}$ and $Q_d \in \mathbb{R}^{K \times K}$ in a row-by-row sequence with an initial step

$$
Q_{c,1} = Q_{c,0}P = \begin{bmatrix}
c_0^T \\
c_1^T \\
\vdots \\
c_{K-1}^T \\
\end{bmatrix}, \quad Q_{d,1} = Q_{d,0}P = \begin{bmatrix}
d_0^T \\
d_1^T \\
\vdots \\
d_{K-1}^T \\
\end{bmatrix}
$$

using the recurrent relations $Q_{c,i+1} = Q_{c,i}P$, $Q_{d,i+1} = Q_{d,i}P$. Hence, any element belonging to the $i+1$-st row can be defined as

$$
q(i+1,l) = \sum_{i=0}^{K} \sum_{i=l}^{K} q(i,)p(i,l)
$$

(A5)

3. Finally, compute both required matrices $C = C_K$ and $D = D_K$ employing the same scheme and starting with

$$
C_1 = C_0Q_c = \begin{bmatrix}
c_0^T \\
c_1^T \\
\vdots \\
c_{KL+1}^T \\
\end{bmatrix}, \quad D_1 = D_0Q_d = \begin{bmatrix}
d_0^T \\
d_1^T \\
\vdots \\
d_{KL+1}^T \\
\end{bmatrix}
$$

(A6)

---

Appendix B

The elements of sub-matrices $S^A_\alpha \in \mathbb{R}^{(KL+1) \times (K+1)}$ and $S^B_\beta \in \mathbb{R}^{(KL+1) \times (K+1)}$ in (20) are found from (11) and (12) as

$$
\frac{1}{A_k} \frac{\partial A_k}{\partial \alpha^*, \beta^*}, \quad i \in \{0, N\}, k \in \{0, NL\},
$$

$$
\frac{1}{B_k} \frac{\partial B_k}{\partial \alpha^*, \beta^*}, \quad i \in \{0, M\}, k \in \{0, ML\}
$$

(B1)

The elements of $S^A_{c,\alpha}$, $S^A_{d,\beta}$, $S^B_{c,\alpha}$, $S^B_{d,\beta} \in \mathbb{R}^{(KL+1) \times (L+1)}$ in (20) are determined from (11) and (12) in a product form as

$$
\begin{bmatrix}
\frac{\partial A_k}{\partial \alpha}, \frac{\partial d_{n-i,k-1}}{\partial \beta}
\end{bmatrix}, \quad \begin{bmatrix}
\frac{\partial A_k}{\partial \alpha}, \frac{\partial d_{n-i,k-1}}{\partial \beta}
\end{bmatrix},
$$

(B2)

$$
\begin{bmatrix}
\frac{\partial B_k}{\partial \alpha}, \frac{\partial d_{n-i,k-1}}{\partial \beta}
\end{bmatrix}, \quad \begin{bmatrix}
\frac{\partial B_k}{\partial \alpha}, \frac{\partial d_{n-i,k-1}}{\partial \beta}
\end{bmatrix}
$$

The elements

$$
\begin{bmatrix}
\frac{\partial \sigma^*}{\partial \sigma}, \frac{\partial \omega^*}{\partial \omega}, \frac{\partial \sigma^*}{\partial \sigma}, \frac{\partial \omega^*}{\partial \omega}
\end{bmatrix}
$$

(B3)

of sub-matrices denoted in (19)

$$
P_\alpha = [P_{\alpha^*:P_{\alpha}}]^T \in \mathbb{R}^{2 \times K}, \quad P_\beta = [P_{\beta^*:P_{\beta}}]^T \in \mathbb{R}^{2 \times K},
$$

$$
Z_\alpha = [Z_{\alpha^*:Z_{\alpha}}]^T \in \mathbb{R}^{2 \times K}, \quad Z_\beta = [Z_{\beta^*:Z_{\beta}}]^T \in \mathbb{R}^{2 \times K}
$$

(B4)

are found from (6) with respect to original poles and zeros.

The elements

$$
\begin{bmatrix}
\frac{\partial \sigma^*}{\partial \sigma}, \frac{\partial \omega^*}{\partial \sigma}, \frac{\partial \sigma^*}{\partial \sigma}, \frac{\partial \omega^*}{\partial \sigma}
\end{bmatrix}, \quad \begin{bmatrix}
\frac{\partial \sigma^*}{\partial \sigma}, \frac{\partial \omega^*}{\partial \sigma}, \frac{\partial \sigma^*}{\partial \sigma}, \frac{\partial \omega^*}{\partial \sigma}
\end{bmatrix}
$$

(B5)

of the sub-matrices denoted in (20)

$$
P_\alpha = [P_{\alpha^*:P_{\alpha}}]^T \in \mathbb{R}^{2 \times (KL+1)}, \quad P_\beta = [P_{\beta^*:P_{\beta}}]^T \in \mathbb{R}^{2 \times (KL+1)},
$$

$$
Z_\alpha = [Z_{\alpha^*:Z_{\alpha}}]^T \in \mathbb{R}^{2 \times (KL+1)}, \quad Z_\beta = [Z_{\beta^*:Z_{\beta}}]^T \in \mathbb{R}^{2 \times (KL+1)}
$$

(B6)

are found differentiating (6) with respect to the mapping parameters $\alpha$ and $\beta$, respectively.

The elements

$$
\begin{bmatrix}
\frac{\partial \alpha^*}{\partial \sigma}, \frac{\partial \beta^*}{\partial \sigma}, \frac{\partial \beta^*}{\partial \sigma}, \frac{\partial \beta^*}{\partial \sigma}
\end{bmatrix}
$$

(B7)

in sub-matrices $P_a \in \mathbb{R}^{(KL+1) \times 2K}$ and $Z_b \in \mathbb{R}^{(KL+1) \times 2K}$ in (19) are evaluated straightforward from expressions (9) and (10). The derivatives denoted $\partial c_{i,j}/\partial \alpha^*$ and
\[ \frac{\partial}{\partial \alpha} \left( a_0 + a_1 z^{-1} + \cdots + a_n z^{-n} \right)^k \left( b_0 + b_1 z^{-1} + \cdots + b_n z^{-n} \right)^{k-1} z^{-r} = \frac{\partial}{\partial \beta} \left( b_0 + b_1 z^{-1} + \cdots + b_n z^{-n} \right)^k \left( a_0 + a_1 z^{-1} + \cdots + a_n z^{-n} \right)^{k-1} z^{-r} \]

The quantities on right-hand sides could be considered as elements of two sets of vectors \( \partial C / \partial \alpha_r \) and \( \partial D / \partial \beta_r \). Each quantity associated with the factor \( z^{-r} \) is embedded into \( p + 1 - s \) successive position. Both sets constitute respectively successive rows of both 3D matrices \( \partial C / \partial \alpha \) and \( \partial D / \partial \beta \). Thereby each layer of them represents itself a matrix constituted in a way similar to this proposed in Appendix A to form matrices C and D.

Taking into account expressions in (B9) one can found simultaneously the elements of vectors \( \partial C / \partial \alpha_r \) and \( \partial D / \partial \beta_r \) if \( \alpha_{r-1} \) and \( \beta_{r-1} \) are multiplied by \( k \) and shifted \( r \) positions right respecting the factor \( z^{-r} \). Implementing such row-by-row computations any element embedded in \( i + 1 - s \) row of layers \( \partial C / \partial \alpha_r \) \((r = 0, \ldots, n)\) and \( \partial D / \partial \beta_r \) \((r = 0, \ldots, n)\) is obtained in terms of a certain previous-row element according to the following rule:

\[ q(i+1, j+r) = iq(i, j) \]

In result, the required matrices are constituted and their \( r-th \) layers have the following form:

\[
\frac{\partial C}{\partial \alpha_r} = \begin{cases} 
0 & 0 \\
0 & 0 \\
0 & 0 \\
\cdots & \cdots \\
0 & 0 \\
\end{cases} \quad \frac{\partial D}{\partial \beta_r} = \begin{cases} 
0 & 0 \\
0 & 0 \\
0 & 0 \\
\cdots & \cdots \\
0 & 0 \\
\end{cases}
\]

References


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Vladimir GEORGIYEV was born in Sofia, Bulgaria, in 1936. He received the M.E. degree in radio-engineering from the Technical University (TU), Sofia in 1960, Ph.D. degree from the Electrotechnical Institute of Leningrad in 1969, DSc degree from TU, Sofia in 1995. Since 1960 he has worked in the industry. In 1964 he joined the Department of Theoretical electrotechnique in TU, Sofia and currently he is a professor. His present interests are in the linear and non-linear circuit theory, switched-capacitor networks and s-z transform applications in circuit design.