GENERALIZED SAMPLING THEOREM FOR BANDPASS SIGNALS

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Abstract

Bandpass time-continuos signals are shown being able to be uniquely expressed in terms of the samples $g_i(nT_V)$ of the impulse responses $g_i(t)$ of m linear time-invariant systems with input f(t) sampled at 1/m Nyquist rate. Various known extensions of the sampling theorem can be regarded as special cases of the resulting generalized sampling expansion of f(t).

Keywords

generalized sampling theorem, non-uniform sampling, *N*th-order sampling, bandpass sampling.

1. Introduction

Methods of sampling and reconstruction of time-continuos signals can be explored in many branches of science. Although the first publications describing an application of the sampling theorem to the solution of telecommunication problems appeared more than sixty years ago, the methods of sampling and reconstruction have belonged to the actual topics even today.

Various sampling theorems for band-limited signals can be regarded as special cases of the Papoulis generalized sampling expansion [1].

This expansion deals with the configuration shown in Fig. 1, where a common input $f(t)$ is led into m linear time-invariant (LTI) pre-filters (channels). The outputs of all the filters are then sampled at $(1/m)$th input Nyquist rate. If a mutual independence of the pre-filters is assumed and if no noise is present in the system, the input signal can be exactly reconstructed by adding the outputs of m LTI post-filters with impulse responses (interpolating functions) $y_1(t)$, $y_2(t)$, ...,$y_m(t)$ to the aggregate of samples $g_i(nT_V)$ where T_V is the sampling period.

Recently, the interest of theoreticians specialized in signal reconstruction was oriented more frequently towards bandpass signals [3], [4] than towards band-limited ones.

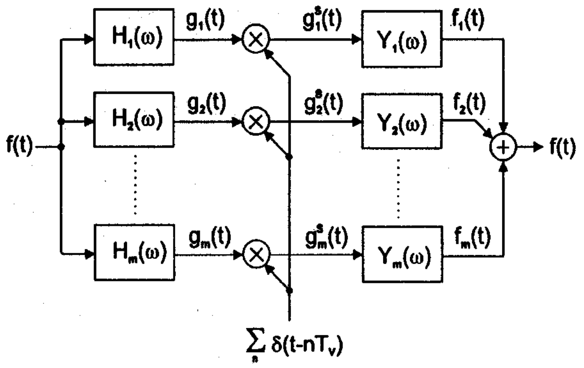

Fig.1 Multi-channel sampling configuration.

In the case of bandpass signals, a final generalization similar to [1] has not been published yet. Its presentation is an aim of this paper.

2. Preliminary Conditions

The bandpass function $f(t)$ with a finite energy (the function occupies L_2) is considered being represented by a finite limit (truncated) inverse Fourier transform, since the spectrum $F(\omega)$ of the function $f(t)$ is assumed being zero outside the band $(-\omega_H, -\omega_D)$ and $(+\omega_D, +\omega_H)$ as it is depicted in Fig 2a.

In the case of higher sampling order, the original spectral components and their replicas are overlapped. The sampling order m has to agree with the number of overlapped spectral replicas, with the number of sub-bands inside the frequency ranges $(+\omega_D, +\omega_H)$ and $(-\omega_H, -\omega_D)$ and with number of linear systems. As described in [5], meeting the above described demands is conditioned by the requirement of m being an even number and by the requirement that the angular frequencies ω_C, $\omega_B=\omega_H-\omega_D$ and $\omega_V=2\pi/T_V$ meet a following condition:

$$\omega_V = \frac{2\omega_C}{k_0 + m/2} \tag{1}$$

where

$$k_0 = m\left(\frac{\omega_C}{\omega_B} - \frac{1}{2}\right) \tag{2}$$

and

$$\omega_V / \omega_B = 2/m \qquad (3)$$

An example of a fourth-order sampled signal spectrum in vicinity of positive and negative original spectral components is shown in Fig. 2a and Fig. 2b.

3. Main Results

Assume m LTI pre-filters with system functions

$$H_1(\omega), H_2(\omega), \ldots, H_m(\omega)$$

If common input bandpass time-continuous signals $f(t)$ are led to the inputs of this system of pre-filters then m resulting output functions are obtained:

$$g_i(t) = \frac{1}{2\pi} \int_{-\omega_H}^{-\omega_D} F(\omega) H_i(\omega) e^{j\omega t} d\omega + \frac{1}{2\pi} \int_{\omega_D}^{\omega_H} F(\omega) H_i(\omega) e^{j\omega t} d\omega \qquad (4)$$

Then, the function $f(t)$ can be expressed in terms of samples $g_i(nT_V)$ of these functions where

$$T_V = m\pi / \omega_B \qquad (5)$$

For this purpose, the following system of equations has to be formed:

$$\underline{H}\,\underline{Y} = \underline{R} \qquad (6)$$

where matrix $\underline{H}$ and vectors $\underline{Y}$, $\underline{R}$ are of the following form

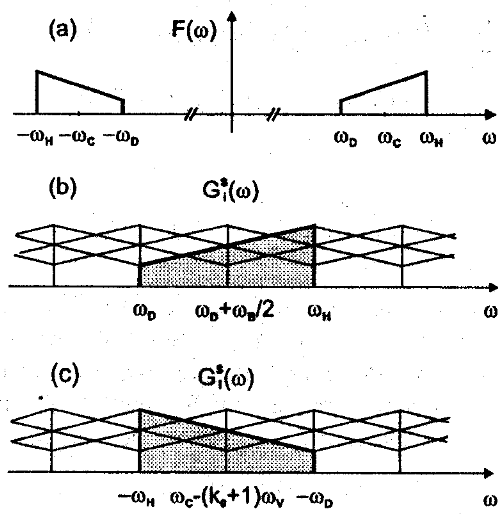

Fig. 2 Spectrum of bandpass signal (a), spectrum of sampled impulse responses of LTI pre-filters in vicinity of positive (b) and negative (c) original spectrum components.

where t is any number and $\omega \in \langle -\omega_H, -\omega_H + \omega_V \rangle$.

This system defines m functions

$$Y_1(\omega,t), Y_2(\omega,t), \ldots, Y_m(\omega,t)$$

of ω ant t because coefficients in matrix $\underline{H}$ depend on ω and the right-hand side depends on t.

The functions $H_i(\omega)$ are general on one hand, but they cannot be entirely arbitrary on the other hand: they must meet the condition that the determinat of the matrix of coefficients differs from zero for every $\omega \in \langle -\omega_H, -\omega_H + \omega_V \rangle$.

$$\underline{H} = \begin{pmatrix} H_1(\omega),, & H_2(\omega), & \ldots & H_m(\omega) \\ H_1(\omega+\omega_V), & H_2(\omega+\omega_V), & \ldots & H_m(\omega+\omega_V) \\ \ldots & \ldots & & \ldots \\ H_1(\omega+(\frac{m}{2}-1)\omega_V), & H_2(\omega+(\frac{m}{2}-1)\omega_V), & \ldots & H_m(\omega+(\frac{m}{2}-1)\omega_V) \\ H_1(\omega+(\frac{m}{2}+k_0)\omega_V), & H_2(\omega+(\frac{m}{2}+k_0)\omega_V), & \ldots & H_m(\omega+(\frac{m}{2}+k_0)\omega_V) \\ H_1(\omega+(\frac{m}{2}+k_0+1)\omega_V), & H_2(\omega+(\frac{m}{2}+k_0+1)\omega_V), & & H_m(\omega+(\frac{m}{2}+k_0+1)\omega_V) \\ \ldots & \ldots & & \ldots \\ H_1(\omega+(k_0+m-1)\omega_V), & H_2(\omega+(k_0+m-1)\omega_V), & \ldots & H_m(\omega+(k_0+m-1)\omega_V) \end{pmatrix} \qquad (7)$$

$$Y = \begin{pmatrix} Y_1(\omega,t) \\ Y_2(\omega,t) \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ Y_{m-1}(\omega,t) \\ Y_m(\omega,t) \end{pmatrix} \qquad R = \begin{pmatrix} 1 \\ \exp(j\omega_V t) \\ \ldots \\ \exp(j(\frac{m}{2}-1)\omega_V t) \\ \exp(j(\frac{m}{2}+k_0)\omega_V t) \\ \exp(j(\frac{m}{2}+k_0+1)\omega_V t) \\ \ldots \\ \exp(j(k_0+m-1)\omega_V t) \end{pmatrix} \qquad (8)$$

The solutions $Y_i(\omega,t)$ can be expanded into a Fourier series in the interval $\omega \in \langle -\omega_H, -\omega_H + \omega_V \rangle$ in which they are defined.

Since the sampled impulse responses $g_i^s(t)$ are of the form

$$g_i^S(t) = g_i(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_V) \qquad (9)$$

the function $f(t)$ at the output of the multi-channel sampling configuration can be described by the following formula

$$f(t)=\sum_{i=1}^{m} g_i^S(t)*y_i(t)=\sum_{n=-\infty}^{\infty}\left[g_1(nT_v)y_1(t-nT_V)+\right. \tag{10}$$
$$\left.+g_2(nT_v)y_2(t-nT_V)+\ldots+g_m(nT_v)y_m(t-nT_V)\right]$$

where

$$y_i(t)=\frac{1}{2\pi}\int_{-\omega_H}^{-\omega_H+\omega_V} Y_i(\omega)e^{j\omega t}d\omega,\quad i=1,\ldots,m \tag{11}$$

The above expressed conclusion can be proven by the similar way as published in [1]. Since the coefficients $H_i(\omega+r\omega_V)$ do not depend on t and since the right-hand side consists of periodic functions (with respect to t) with the period T_V[1], the functions $Y_i(\omega,t)$ have to periodic too

$$Y_i(\omega,t+T_V)=Y_i(\omega,t) \tag{12}$$

On the basis of (11) and (12), the following formula can be obtained

$$y_i(t-nT_V)=\frac{1}{2\pi}\int_{-\omega_H}^{-\omega_H+\omega_V} Y_i(\omega,t-nT_V)e^{j\omega(t-nT_V)}d\omega= \tag{13}$$
$$=\frac{1}{2\pi}\int_{-\omega_H}^{-\omega_H+\omega_V} Y_i(\omega,t)e^{-j\omega nT_V}e^{j\omega t}d\omega.$$

The eqn. (13) shows that $y_i(t-nT_V)$ is nth coefficient of the Fourier series of the function $Y_i(\omega,t)e^{j\omega t}$ in the frequency range $\omega\in\langle-\omega_H, -\omega_H+\omega_V\rangle$. Therefore

$$e^{j\omega t}Y_i(\omega,T_V)=\sum_{n=-\infty}^{\infty} y_i(t-nT_V)e^{jnT_V\omega} \tag{14}$$

Multiplying the first equation of (6) by $e^{j\omega t}$ and using (14) yields

$$e^{j\omega t}=H_1(\omega)\sum_{n=-\infty}^{\infty} y_i(t-nT_V)e^{jnT_V\omega}+\ldots \tag{15}$$
$$+H_m(\omega)\sum_{n=-\infty}^{\infty} y_m(t-nT_V)e^{jnT_V\omega}$$

Obviously, (15) holds for every $\omega\in\langle-\omega_H, -\omega_D\rangle$ and for $\omega\in\langle\omega_D, \omega_H\rangle$. Indeed, multiplying the rth eqn. of (6) by $e^{j\omega t}$ and using (14), following conclusion can be done for every $\omega\in\langle-\omega_H, -\omega_H+\omega_V\rangle$

$$e^{j(\omega+r\omega_V)t}=H_1(\omega+r\omega_V)\sum_{n=-\infty}^{\infty} y_i(t-nT_V)e^{jnT_V(\omega+r\omega_V)}+\ldots$$
$$+H_m(\omega+r\omega_V)\sum_{n=-\infty}^{\infty} y_m(t-nT_V)e^{jnT_V(\omega+r\omega_V)}$$

[1] The described periodicity comes from the relation $r\omega_V(t+T_V)=r\omega_V t+2\pi r$.

because $e^{jnT(\omega+r\omega_V)}=e^{jnT_V\omega}$. However, if ω varies in the interval $\omega\in\langle-\omega_H, -\omega_H+\omega_V\rangle$ then $\omega+r\omega_V$ varies in the interval $\omega\in\langle-\omega_H+(r-1)\omega_V, -\omega_H+r\omega_V\rangle$, and therefore, eqn. (15) is valid in this interval.

In order to complete the proof of (10) and (11), eqn. (15) is inserted into the formula

$$f(t)=\frac{1}{2\pi}\int_{-\omega_H}^{-\omega_D} F(\omega)e^{j\omega t}d\omega+\frac{1}{2\pi}\int_{\omega_D}^{\omega_H} F(\omega)e^{j\omega t}d\omega \tag{16}$$

Assuming $k_0=0$ and substituting this value into (2), the equality $\omega_C=\omega_B/2$ is obtained. This means, that the bandpass function turns to the band-limited function with the cut-off frequency ω_B and above described sampling theorem turns to the generalized sampling expansion [1]. Therefore, [1] can be said being a special case of (6), (10) and (11).

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Aleš PROKEŠ was born in Znojmo in 1963. He received the M.Sc. degree in Electrical Engineering from the Technical University of Brno in 1988. At the present time, he is studying towards the PhD degree under supervision of Prof. Vladimír Šebesta at the Institute of Radio Electronics at the Technical University of Brno. He is interested in signal processing and optical communication.

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