Generalized Canonical State Models of Third-Order Piecewise-Linear Dynamical Systems and Their Applications

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Abstract

The elementary canonical state models of the third-order autonomous dynamical systems, topologically conjugate to Chua's circuit family, are generalized for any continuous and odd symmetrical piecewise-linear (PWL) feedback function. Their state equations are in accordance with the basic form of the Lur'e systems and the corresponding circuit model contains the multiple PWL feedback. The general results are applied for the simplest three-region case defined by three sets of the equivalent eigenvalue parameters. The application of these results is demonstrated on the double-scroll chaotic attractor with global attracting properties. As an example its utilization in synchronized chaos is shown.

Keywords

Dynamical systems, piecewise-linear modelling, state models, canonical forms, synchronization, chaos, chaotic-masked communication

1. Introduction

The recently developed elementary canonical state models of the third order autonomous dynamical systems [1] can easily be generalized for any continuous and odd symmetrical piecewise-linear (*PWL*) feedback function [2]. Such a general *PWL* function partitions $\Re^3$ by several pairs of parallel planes into several symmetrical regions having different associated characteristic polynomials determined by the so called equivalent eigenvalue parameters [3]. If this model is to be elementary canonical, i.e. with simple relations between its state equation parameters and the equivalent eigenvalue parameters of the individual regions, only the first form described in [1] can be used. In this case its general matrix state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\,h(\mathbf{w}^T\mathbf{x}) \tag{1}$$

is extended to the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{i=1}^{k-1} \mathbf{b}_i\, h_i(\mathbf{w}^T\mathbf{x}) \tag{2}$$

where $\mathbf{A} \in \Re^{3\times3}$, $\mathbf{b}_i \in \Re^3$, $\mathbf{w} \in \Re^3$. It corresponds to the basic form of the Lur'e systems [4] where simple memoryless partial *PWL* functions

$$h_i(\mathbf{w}^T\mathbf{x}) = \tfrac{1}{2}\left(\left|\mathbf{w}^T\mathbf{x}+E_i\right| - \left|\mathbf{w}^T\mathbf{x}-E_i\right|\right),\ i = 1,2,\ldots,k-1; \tag{3}$$

represent elementary nonlinearities while all linear relations are included into state matrix $\mathbf{A}$.

For an illustration the simplest case with k=3 (i=1,2) is considered. Then the *PWL* feedback function partitions $\Re^3$ by two pairs of parallel planes, i.e.

$$U_{+1}:\ \mathbf{w}^T\mathbf{x} = E_1,\quad U_{-1}:\ \mathbf{w}^T\mathbf{x} = -E_1,$$

and $$U_{+2}:\ \mathbf{w}^T\mathbf{x} = E_2,\quad U_{-2}:\ \mathbf{w}^T\mathbf{x} = -E_2,$$

into the following regions:

(a) one central inner region $D_0\ (-E_1 \le \mathbf{w}^T\mathbf{x} \le E_1)$,

(b) two symmetrical inner regions $(E_2 > E_1 > 0)$

$$D_{+1}\ (E_1 \le \mathbf{w}^T\mathbf{x} \le E_2),\ D_{-1}\ (-E_2 \le \mathbf{w}^T\mathbf{x} \le -E_1),$$

(c) two symmetrical outer regions

$$D_{+2}\ (\mathbf{w}^T\mathbf{x} \ge E_2),\quad D_{-2}\ (\mathbf{w}^T\mathbf{x} \le -E_2)$$

For special case $\mathbf{w}^T\mathbf{x} = x_1$ it is shown in Fig. 1a,2,3a.

The dynamical behaviour of this system is determined by the characteristic polynomials associated with the individual regions

(a) D_0: $$P(s) = \det(s\mathbf{1}-\mathbf{A}_0) = (s-\mu_1)(s-\mu_2)(s-\mu_3) = s^3 - p_1 s^2 + p_2 s - p_3, \tag{4}$$

(b) D_{+1}, D_{-1}: $$Q(s) = \det(s\mathbf{1}-\mathbf{A}_1) = (s-\nu_1)(s-\nu_2)(s-\nu_3) = s^3 - q_1 s^2 + q_2 s - q_3, \tag{5}$$

(c) D_{+2}, D_{-2}: $$R(s) = \det(s\mathbf{1}-\mathbf{A}) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3) = s^3 - r_1 s^2 + r_2 s - r_3 \tag{6}$$

where $\mathbf{1}$ is the unity matrix. The individual state matrices in eqns (4),(5),(6) are mutually related by the formulas

$$\mathbf{A}_1 = \mathbf{A} + \mathbf{b}_2\mathbf{w}^T, \tag{7}$$

$$\mathbf{A}_0 = \mathbf{A}_1 + \mathbf{b}_1\mathbf{w}^T = \mathbf{A} + (\mathbf{b}_1 + \mathbf{b}_2)\mathbf{w}^T \tag{8}$$

2. Basic Canonical State Model

If the state model is to be elementary canonical [1] it can be proved that state matrix $\mathbf{A}$ and vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{w}$ must have the following forms:

$$\mathbf{A} = \begin{bmatrix} r_1 & -1 & 0 \\ r_2 & 0 & -1 \\ r_3 & 0 & 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad (9a,b)$$

$$\mathbf{b}_1 = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} q_1 - r_1 \\ q_2 - r_2 \\ q_3 - r_3 \end{bmatrix}, \qquad (9c,d)$$

i.e. the complete state equations are

$$\frac{dx_1}{dt} = r_1x_1 - x_2 + (p_1-q_1)h_1(x_1) + (q_1-r_1)h_2(x_1), \qquad (10a)$$

$$\frac{dx_2}{dt} = r_2x_1 - x_3 + (p_2-q_2)h_1(x_1) + (q_2-r_2)h_2(x_1), \qquad (10b)$$

$$\frac{dx_3}{dt} = r_3x_1 + (p_3-q_3)h_1(x_1) + (q_3-r_3)h_2(x_1) \qquad (10c)$$

The partial *PWL* feedback functions and the individual regions are illustrated in Fig.1a, the integrator-based circuit model corresponding to eqns (10a,b,c) is shown in Fig.1b. It consists of three ideal integrators, three basic adders, and a double feedback realized by two blocks defined by partial *PWL* functions $h_1(x_1)$, $h_2(x_1)$.

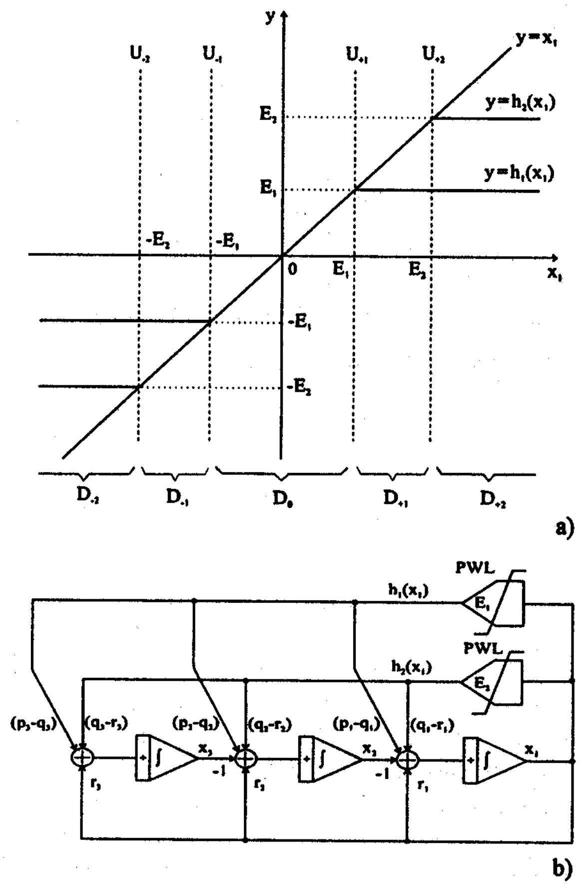

Fig. 1. Basic elementary canonical state model. a) Partial *PWL* feedback functions and the individual regions in eqns (10), b) Integrator-based circuit model

3. Modified State Model

By rearranging eqns (10a,b,c) into the form [1] the following modified expressions are obtained

$$\frac{dx_1}{dt} = p_1h_1(x_1) + q_1[h_2(x_1) - h_1(x_1)] + r_1[x_1 - h_2(x_1)] - x_2, \qquad (11a)$$

$$\frac{dx_2}{dt} = p_2h_1(x_1) + q_1[h_2(x_1) - h_1(x_1)] + r_2[x_1 - h_2(x_1)] - x_3, \qquad (11b)$$

$$\frac{dx_3}{dt} = p_3h_1(x_1) + q_3[h_2(x_1) - h_1(x_1)] + r_3[x_1 - h_2(x_1)] \qquad (11c)$$

The partial *PWL* feedback functions are illustrated in Fig. 2a,b,c. The corresponding modified integrator-based circuit model containing two additional adders is shown in Fig. 2d. In this structure the basic adder gains are directly determined by equivalent eigenvalue parameters p_j, q_j, r_j.

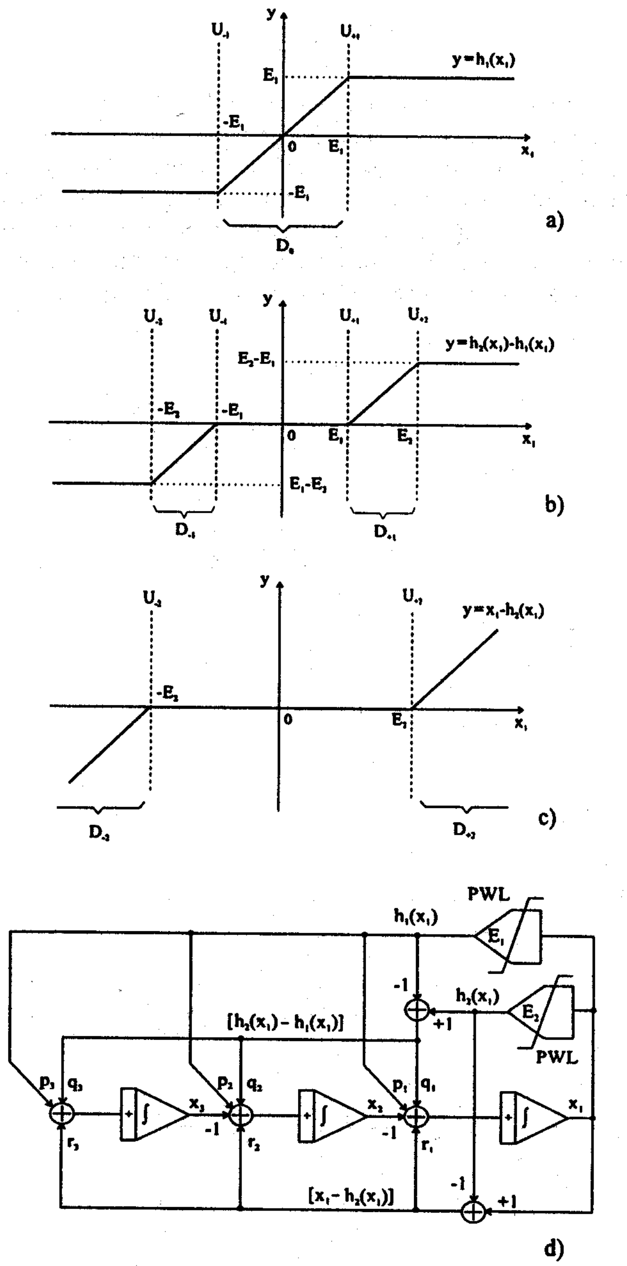

Fig. 2. Modified elementary canonical state model. a), b), c) Partial *PWL* feedback functions and the individual regions in eqns (11), d) Integrator-based circuit model

4. Global State Model

Denoting in eqns (10a,b,c)

$$f_1(x_1) = r_1x_1+(p_1-q_1)\,h_1(x_1)+(q_1-r_1)\,h_2(x_1)\ , \quad (12a)$$

$$f_2(x_1) = r_2x_1+(p_2-q_2)\,h_1(x_1)+(q_2-r_2)\,h_2(x_1)\ , \quad (12b)$$

$$f_3(x_1) = r_3x_1+(p_3-q_3)\,h_1(x_1)+(q_3-r_3)\,h_2(x_1) \quad (12c)$$

the state equations can be rewritten into the global form

$$\frac{dx_1}{dt} = -x_2 + f_1(x_1)\ , \quad (13a)$$

$$\frac{dx_2}{dt} = -x_3 + f_2(x_1)\ , \quad (13b)$$

$$\frac{dx_3}{dt} = f_3(x_1) \quad (13c)$$

The first partial function corresponding to eqn (12a) is illustrated in Fig. 3a. The integrator-based circuit model corresponding to the global state equations (13a,b,c) contains only two adders and three PWL feedback blocks (Fig. 3b) having the partial transfer functions defined by eqns (12a,b,c) each of them being represented by the triplet of related equivalent eigenvalue parameters p_j, q_j, r_j.

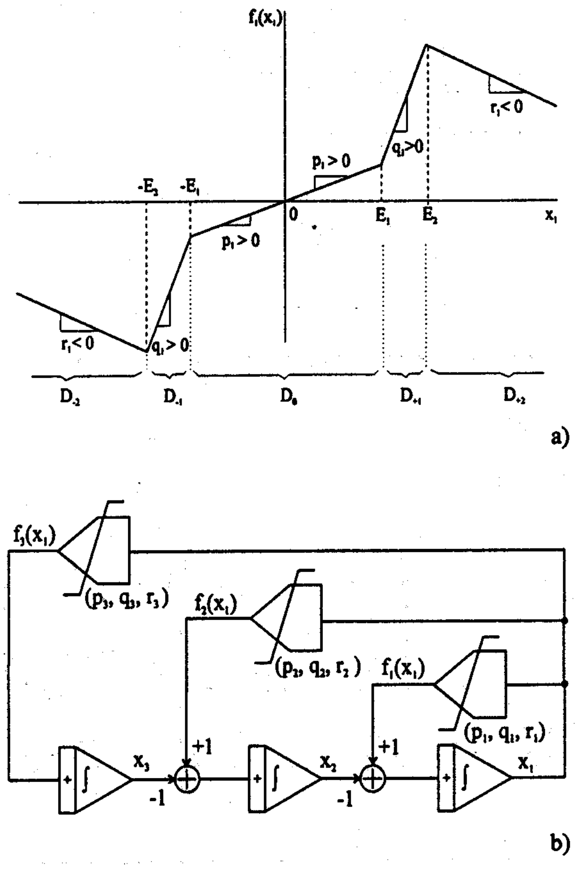

Fig. 3. Global state model a) Illustration of the first partial *PWL* feedback function corresponding to eqn (12a), b) integrator-based circuit model corresponding to eqns (13).

5. Application in synchronized chaos

As an illustrative example of the model derived a chaotic attractor with the global attracting set, i.e. a chaotic motion starts up for any initial conditions, is presented. One of the serious problems in practical realization of Chua's circuit family (i.e. symmetrical three region PWL dynamical systems) is the fact that the attracting set (basin) is limited. If the state vector gets out of this basin the system exhibits unbounded motion. It is true for all the attractors from Chua's family because at least one eigenvalue has positive real part in the outer regions of the PWL function. When the state vector is large enough then the influence of the middle region becomes negligible to the system dynamics and an „almost" linear unstable system is obtained. In a real circuit the supply voltage of active network elements (op amps in the most cases) limits the unbounded motion. The model derived is suitable for both the analysis of the phenomenon and synthesis of new chaotic systems exhibiting a global attracting set.

The new system is derived from the original one by other two additional regions D+2 and D-2 in the state space (Fig. 1a). The necessary condition for a global attraction property is the negative real part of all eigenvalues in the regions, i.e. $Re\,(\lambda_i) < 0$ in eqn (6). Additionally, the system must possess an operating point neither in D_{+2} nor in D_{-2} because the stable point represents a trap preventing oscillations if the state vector hits the attracting region. The operating point $\mathbf{x^*}$ can be obtained from eqn (2) considering $\dot{\mathbf{x}} = 0$, i.e.

$$\mathbf{0} = \mathbf{A}\,\mathbf{x}^* + \mathbf{b}_1\,h_1(\mathbf{w}^T\mathbf{x}^*) + \mathbf{b}_2\,h_2(\mathbf{w}^T\mathbf{x}^*) \quad (14)$$

and satisfying the inequality

$$\left|\mathbf{w}^T\mathbf{x}^*\right| < E_2 \quad (15)$$

This condition can easily be expressed for the basic state model given by eqns (9) where relation (15) is reduced to $|x_1^*| < E_2$. Substituting x_1^* from the solution of eqn (14) the final form of the condition is obtained

$$E_2\frac{q_3}{r_3} > E_1\frac{q_3-p_3}{r_3} \quad (16)$$

As an example the well-known double-scroll attractor has been used [3]. For this case $E_1 = 1$ and the eigenvalues of the additional region are chosen $\lambda_1 = \lambda_2 = \lambda_3 = -1$. Then the suitable value of E_2 must be in the interval $(2.2 \div 2.7)$ where the condition (16) is automatically satisfied and the numerical simulation shows that outside this interval the system exhibits a stable periodic orbit. The chaotic attractors for $E_2 = 2.5$ and two different sets of initial conditions are presented in Fig. 7a,b.

The global attraction property can be useful for various concepts of chaotic-masked communication systems where the transmitter is modulated with an information signal. It assures that the transmitter can recover from any input signal, whereas the original three-region systems can get in saturation or periodic motion without a possibility of recovering from this state. The block diagram of a simple synchronized system derived from the elementary canonical models of Chua's circuit family is shown in Fig. 8 [5].

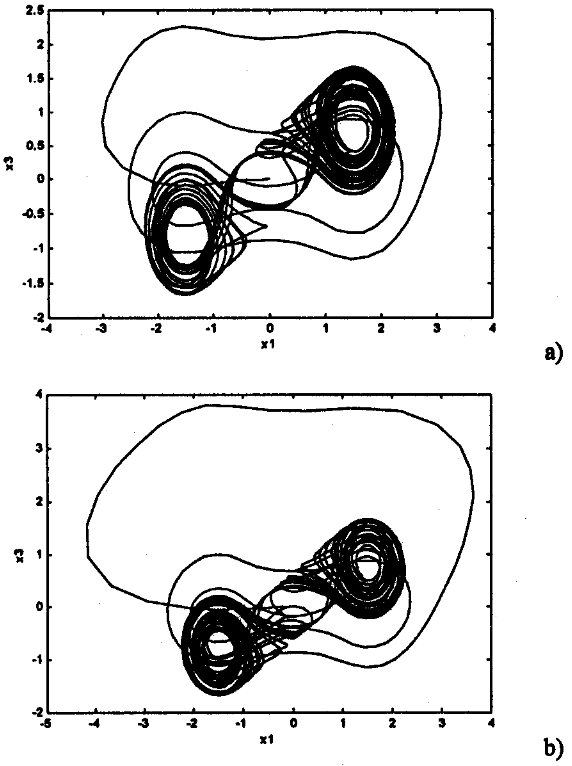

Fig. 7. Double-scroll attractor for two sets of initial conditions a) $x_1 = x_3 = 0$, $x_2 = 5.0$; b) $x_1 = x_3 = 0$, $x_2 = 10.0$; (p_1=0.09, p_2=0.432961, p_3=0.653325; q_1= -1.168, q_2=0.846341, q_3= -1.2948; r_1 = -3.0, r_2 = 3.0, r_3= -1.0;)

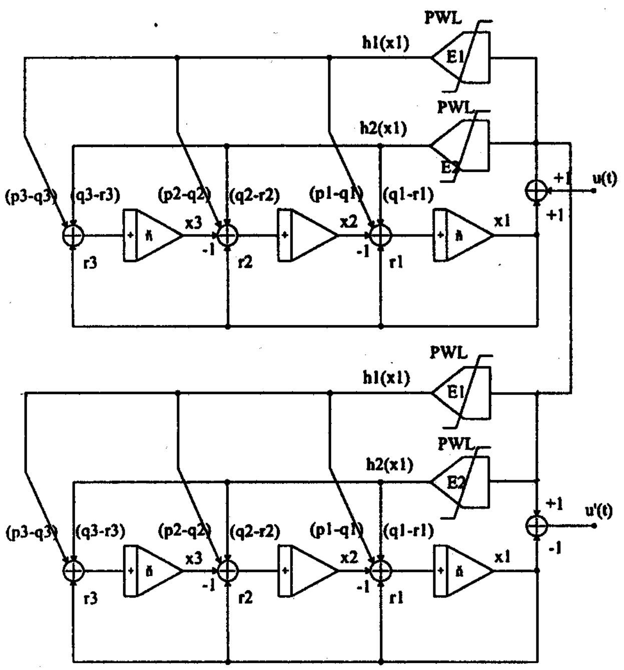

Fig. 8. Simple synchronized system based on the elementary canonical models of Chua's circuit family and its utilization in the chaotic-masked communication

6. Conclusion

The results presented can easily be extended for any k>3 and also for n-dimensional Lur'e systems [4]. The generalized elementary canonical state models represent a suitable tool for the modelling and simulation of the chaotic phenomena in dynamical *PWL* systems as shown in the example of the chaotic attractor with the global attracting and synchronized chaos simulation. They can also be used as prototypes for the practical realization of the corresponding circuit models [6].

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