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Mutual Relations Between Multiple-Input Linear and Multiple-Feedback Piecewise-Linear Dynamical Systems

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Abstract

State models of multiple-input linear non-autonomous dynamical systems are modified to the autonomous form using a multiple piecewise-linear (PWL) feedback. By such a way the corresponding state models of PWL autonomous dynamical systems topologically conjugate to Lur'e systems [4] can be derived. As an example, the canonical state models of multiple-input linear and the corresponding multiple PWL feedback systems are proposed. Their state matrix equations and the integrator-based circuit models are shown.

Keywords

Dynamical systems, Linear systems, Piecewise-linear systems, State models, Canonical forms

1. Introduction

The paper gives the new generalized viewpoint to the relation between *linear non-autonomous* and *piecewise-linear (PWL) autonomous* dynamical systems. This problem has particularly been solved in [2] for *single-input single-output linear* and corresponding *single feedback PWL* systems. In such a case the linear system is described by the state equations [1] in the matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}v \tag{1}$$

$$y = \mathbf{C}\mathbf{x} + \mathrm{D}v \tag{2}$$

where $\mathbf{x}$ is the vector of the state variables, v and y is the input and output variables, respectively. Using the Laplace transform the transfer function of the nonautonomous linear system is generally given [2] as

$$K(s) = \frac{Y(s)}{V(s)} = \mathrm{D} + \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} = \ldots = \mathrm{D}\frac{P(s)}{Q(s)} \tag{3}$$

where $\mathbf{1}$ is the unity matrix.

The corresponding autonomous PWL dynamical system is described by the modified state equation (1), i.e.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\,h(\mathbf{C}\mathbf{x}) \tag{4}$$

where the input variable v is replaced by simple continuous and odd-symmetric PWL function

$$v = h(\mathbf{C}\mathbf{x}) = \frac{1}{2}\left(\left|\mathbf{C}\mathbf{x}+1\right| - \left|\mathbf{C}\mathbf{x}-1\right|\right) \tag{5}$$

partitioning $\Re^3$ by two parallel planes U_{+1}: $\mathbf{C}\mathbf{x} = +1$, U_{-1}: $\mathbf{C}\mathbf{x} = -1$ into inner region D_0: $(-1 \le \mathbf{C}\mathbf{x} \le +1)$ and two outer regions D_{+1}: $(\mathbf{C}\mathbf{x} \ge +1)$, D_{-1}: $(\mathbf{C}\mathbf{x} \le -1)$ - (Fig. 1).

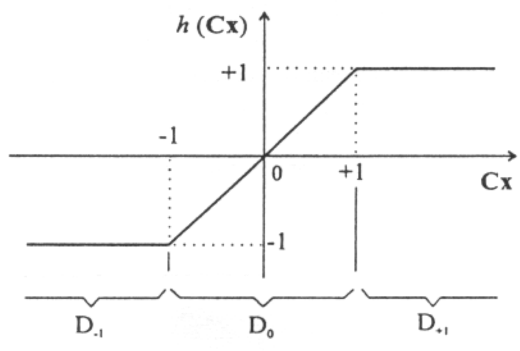


Fig. 1. Simple memoryless PWL feedback function

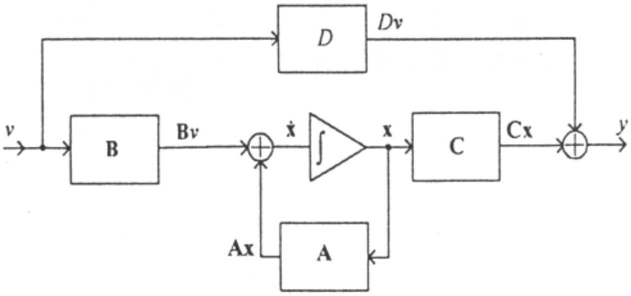
Dynamical behavior of this system is determined by two characteristic polynomials $Q(s)$, $P(s)$ associated with the individual regions. They are identical to polynomials $Q(s)$ and $P(s)$ in (3) and given by the general formulas [2]

$$Q(s) = \det(s\mathbf{1} - \mathbf{A}) \tag{6}$$

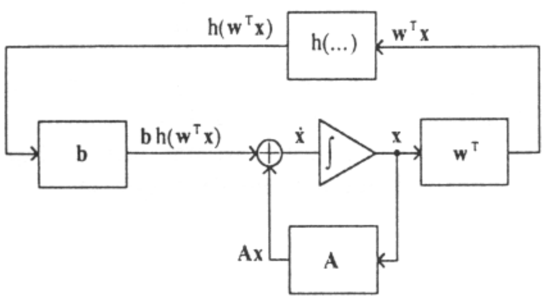
$$P(s) = \det(s\mathbf{1} - \mathbf{A}_0) \tag{7}$$

where $$\mathbf{A}_0 = \mathbf{A} - \mathbf{B}\mathbf{C}\mathrm{D}^{-1} \tag{8}$$

The results applied originally only to the elementary canonical state models [1] have been extended also for other state models topologically conjugate to Class C [5]. The corresponding general block diagrams are shown in Fig. 2a,b.



(a)



(b)

Fig. 2. General block diagrams for linear system with single input (a), and the corresponding system with simple PWL feedback (b)

2. Generalized results

Utilizing the results presented in [3] the solution [2] can be extended also for the *multiple-input* linear and *multiple-feedback* PWL systems. Then the previous formulas are generalized into the following forms:

2.1 Non-autonomous multiple-input linear systems

In this case (Fig. 3a) the general state equation matrix form is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v} \tag{1a}$$

$$y = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{v} \tag{2a}$$

where $\mathbf{B}=[\mathbf{B}_1 \dots \mathbf{B}_k]$, $\mathbf{D}=[D_1 \dots D_k]$, $\mathbf{v}^T=[v_1 \dots v_k]$.

The output variable represents the linear combination of all input variables, i.e.

$$Y(s) = \mathbf{K}(s)\mathbf{V}(s) = \sum K_i(s)V_i(s) \tag{3}$$

and the corresponding transfer matrix can be expressed as

$$\mathbf{K}(s) = \mathbf{D} + \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} = [K_1(s) \dots K_k(s)] \tag{3a}$$

2.2 Autonomous multiple-feedback piecewise-linear systems

In the corresponding autonomous systems (Fig. 3b) the general state equations are extended into the form corresponding to the Lur'e systems [4], i.e.

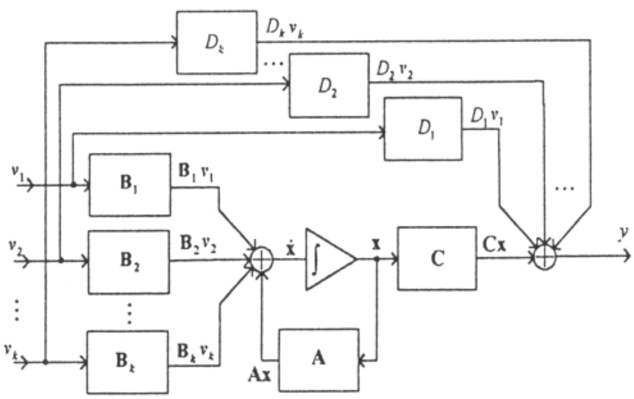
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum \mathbf{B}_i h_i(\mathbf{C}\mathbf{x}) \ , \quad (i = 1 \text{ to } k) \tag{4a}$$

where particular PWL feedback functions represent, in accordance with eqn (5), the elementary nonlinearities

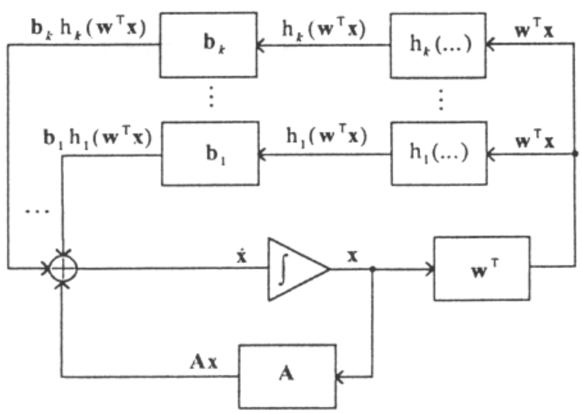
$$h_i(\mathbf{C}\mathbf{x}) = \tfrac{1}{2}\left(\left|\mathbf{C}\mathbf{x} + E_i\right| - \left|\mathbf{C}\mathbf{x} - E_i\right|\right) \tag{5a}$$

partitioning $\Re^3$ by k pairs of parallel planes into one central inner region, (k-1) pairs of symmetric inner regions, and one pair of symmetric outer regions.

Characteristic polynomials associated with inner regions are the same as those in numerators of the individual particular transfer functions $K_i(s)$ of the corresponding non-autonomous linear system, while the characteristic polynomial of outer regions represents the common denominator of all these transfer functions.



(a)



(b)

Fig. 3. General block diagrams for linear system with multiple input (a), and the corresponding system with multiple PWL feedback (b)

3. Examples

The detailed form of the above mentioned relations can easily be demonstrated for the simplest case $k = 2$ (two-input linear and double-feedback PWL systems), where

$$\mathbf{B}=[\mathbf{B}_1\ \ \mathbf{B}_2],\ \ \mathbf{D}=[D_1\ \ D_2],\ \ \mathbf{v}^T=[v_1\ \ v_2]\,.$$

Then the input-output relation of the non-autonomous linear system is expressed in the form

$$Y(s)=K_1(s)V_1(s)+K_2(s)V_2(s) \qquad (3b)$$

and the particular transfer functions can be written as

$$K_1(s)=D_1\frac{P(s)}{R(s)}\,, \qquad K_2(s)=D_2\frac{Q(s)}{R(s)}\,,$$

where the individual polynomials are

$$P(s)=\det(\mathbf{s}\mathbf{1}-\mathbf{A}_0)=$$
$$=s^3-p_1s^{n-1}+p_2s^{n-2}-\ldots+(-1)^{n+1}p_{n-1}s+(-1)^n p_n$$

$$Q\ (s)=\det(\mathbf{s}\mathbf{1}-\mathbf{A}_1)=$$
$$=s^3-q_1s^{n-1}+q_2s^{n-2}-\ \ldots+(-1)^{n+1}q_{n-1}s+(-1)^n q_n$$

$$R(s)=\det(\mathbf{s}\mathbf{1}-\mathbf{A})=$$
$$=s^3-q_1s^{n-1}+q_2s^{n-2}-\ \ldots+(-1)^{n+1}q_{n-1}s+(-1)^n q_n$$

Choosing $\mathbf{D}=[-1\ \ -1]$ the particular state matrices are

$$\mathbf{A}_1=\mathbf{A}+\mathbf{B}_2\mathbf{C} \quad \text{and} \quad \mathbf{A}_0=\mathbf{A}_1+\mathbf{B}_1\mathbf{C}$$

The corresponding PWL autonomous double-feedback system is described by the matrix form

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}_1\,h_1(\mathbf{C}\mathbf{x})+\mathbf{B}_2\,h_2(\mathbf{C}\mathbf{x})$$

where the individual PWL functions are given by eqn (5a) with the breakpoint values $E_2>E_1>0$. Then the PWL feedback functions partition $\Re^3$ by two pairs of parallel planes, i.e.

$$U_{+1}:\ \mathbf{C}\mathbf{x}=E_1\,,\quad U_{-1}:\ \mathbf{C}\mathbf{x}=-E_1\,,$$

and $$U_{+2}:\ \mathbf{C}\mathbf{x}=E_2\,,\quad U_{-2}:\ \mathbf{C}\mathbf{x}=-E_2\,,$$

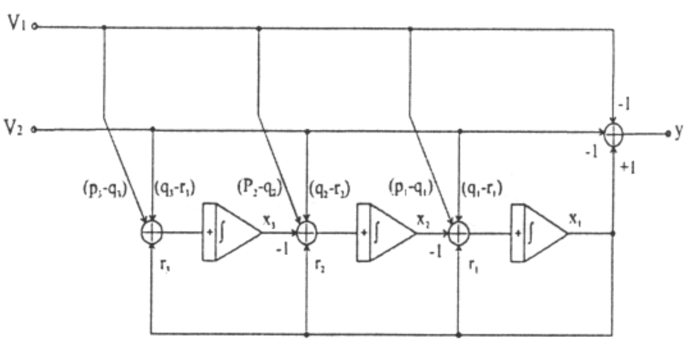
into the one central inner region D_0: $(-E_1\le\mathbf{C}\mathbf{x}\le E_1)$, two symmetrical inner regions D_{+1}: $(E_1\le\mathbf{C}\mathbf{x}\le E_2)$, $D_{-1}(-E_2\le\mathbf{C}\mathbf{x}\le -E_1)$, and two symmetrical outer regions D_{+2}: $(\mathbf{C}\mathbf{x}\ge E_2)$, D_{-2}: $(\mathbf{C}\mathbf{x}\le -E_2)$ with associated characteristic polynomials $P(s)$, $Q(s)$, and $R(s)$, respectively.

Consider the first elementary canonical state models of the linear and piecewise-linear dynamical systems [1],[2] where the state matrix and all vectors have the simple forms

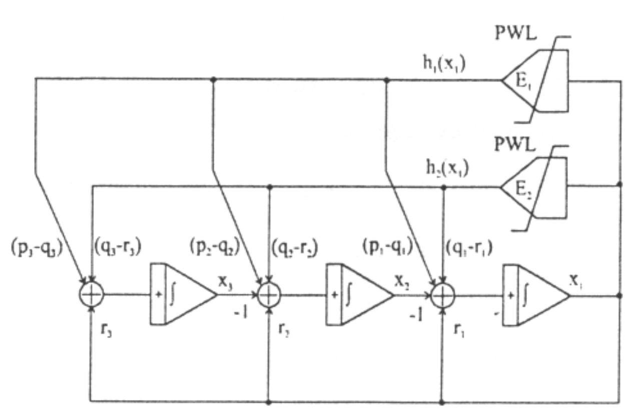
$$\mathbf{A}=\begin{bmatrix} r_1 & -1 & 0 & 0 & \ldots & 0\\ r_2 & 0 & -1 & 0 & \ldots & 0\\ r_3 & 0 & 0 & -1 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & . & \vdots\\ r_{n-1} & 0 & 0 & 0 & \ldots & -1\\ r_n & 0 & 0 & 0 & \ldots & 0 \end{bmatrix},\ \mathbf{C}^T=\begin{bmatrix}1\\0\\0\\ \vdots\\0\\0\end{bmatrix}$$

$$\mathbf{B}_1=\begin{bmatrix} p_1-q_1\\ p_2-q_2\\ p_3-q_3\\ \vdots\\ p_{n-1}-q_{n-1}\\ p_n-q_n \end{bmatrix},\quad \mathbf{B}_2=\begin{bmatrix} q_1-r_1\\ q_2-r_2\\ q_3-r_3\\ \vdots\\ q_{n-1}-r_{n-1}\\ q_n-r_n \end{bmatrix}$$

The corresponding integrator-based circuit models for the case of the third-order systems are shown in Fig. 4a (two-input non-autonomous linear system) and in Fig. 4b (double-feedback autonomous PWL system).



(a)



(b)

Fig. 4. Integrator-based circuit models of the 3-D dynamical system - (first elementary canonical forms). (a) Two-input linear non-autonomous system, (b) double-feedback piecewise-linear autonomous system

4. Conclusion

The generalized relation between canonical state models of non-autonomous multiple-input linear and autonomous multiple-feedback piecewise-linear dynamical systems is shown. The results introduced can be utilized in two following cases:

(i) Starting from the multiple-input linear prototype (not necessary stable) possessing the set of the known transfer function, the corresponding state model of multiple-feedback PWL dynamical system can easily be derived. Here characteristic polynomials associated with inner regions of PWL system are the same as those in numerators of the individual particular transfer functions $K_i(s)$ of the linear system and the characteristic polynomial of outer regions represents the common denominator of all these transfer functions.

(ii) Polynomials in numerators of the individual particular transfer functions of multiple-input linear system can easily be expressed by simple procedure (6) based on the state matrix eigenvalues definition utilized for common denominator. Choosing matrix $\mathbf{D}=[-1 \;\ldots\; -1]$, the corresponding modified state matrices can be calculated by formula, which is formally the same as (8) currently used for characteristic polynomials of PWL systems.

Using the linear topological conjugacy [4] all the known canonical state models of nonautonomous linear system, as well as other new state models [5], can easily be expressed by the qualitatively equivalent elementary canonical form [1], including Chua's model [7]. The utilization of new models with multiple PWL feedback is introduced in [6]. Other details will be discussed in the contribution.

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