

THE MODIFIED NODE VOLTAGE METHOD AS TOOL FOR CALCULATION OF NETWORK FUNCTIONS AND SENSITIVITIES

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Abstract

The paper deals with the well-known modified node voltage method that is generally used for formulation of system equations in universal network simulation programs. It shows very efficient ways if utilizing this type of equation formulation for evaluation of network transfer functions and their sensitivities to variations of element parameters. The described procedures can be used very effectively namely in connection with some matrix oriented mathematical program like for instance MATLAB. Several illustrative examples support the explanation of the methods described in the paper.

Keywords

network function, sensitivity, MATLAB

1. Introduction

The modified node voltage method [1] has proved itself to be very effective for formulation of network equations for complicated linear and linearized electrical and electronic networks. The method is able to treat all basic network elements including for instance ideal voltage sources controlled or uncontrolled, current controlled sources, ideal operational amplifiers, zero value resistances and coils at zero frequency. The equation formulation may be simply algorithmized and system equations can be set up automatically following the circuit description in netlist (a list of elements, their connection to network nodes, and parameters).

Network simulation programs usually treat the simulated network as a two-port with defined input and output terminals, input signal source (current I_m or voltage U_{in}) and output signal sensor (current I_{out} or voltage U_{out}). Ratio of output and input signals determines the network function that is to be calculated:

- transfer impedance (output open-circuited)

$$Z_T = \frac{U_{out}}{I_{in}}, \quad (1)$$

- current transfer factor (output short-circuited)

$$K_i = \frac{I_{out}}{I_{in}}, \quad (2)$$

- voltage transfer factor (output open-circuited)

$$K_u = \frac{U_{out}}{U_{in}}, \quad (3)$$

- transfer admittance (output short-circuited)

$$Y_T = \frac{I_{out}}{U_{in}}. \quad (4)$$

The network to be solved has n_u independent nodes (plus one reference node with zero potential), that means n_u unknown node voltages and n_i additional unknown currents that have been chosen to complement the whole system and make it solvable. Thus the total number of unknown variables is $n_n = n_u + n_i$.

The resultant system equations resemble the hybrid (parallel-series) equations of a two-port and have the form

$$\mathbf{Kx} = \mathbf{b}, \quad (5)$$

where the system matrix

$$\mathbf{K} = \begin{bmatrix} \mathbf{Y} & \mathbf{B} \\ \mathbf{A} & \mathbf{Z} \end{bmatrix} \quad (6)$$

consists of 4 sub-matrices:

- $\mathbf{Y}(1:n_u, 1:n_u)$ is a classical square matrix of admittances,
- $\mathbf{B}(1:n_u, 1:n_i)$ is a generally rectangular dimensionless matrix of current relations,
- $\mathbf{A}(1:n_i, 1:n_u)$ is a generally rectangular dimensionless matrix of voltage relations,
- $\mathbf{Z}(1:n_i, 1:n_i)$ is a square matrix of impedances.

Further in (5), $\mathbf{x}(1:n_n)$ is vector of n_u node voltages and n_i additional currents, and $\mathbf{b}(1:n_n)$ denotes vector of n_i currents and n_u voltages of independent sources.

Note: When the simulated network is reciprocal then the sub-matrices \mathbf{Y} and \mathbf{Z} are symmetrical with respect to their main diagonal, i.e.

$$\mathbf{Y} = \mathbf{Y}^T, \quad \mathbf{Z} = \mathbf{Z}^T. \quad (7)$$

The sub-matrices \mathbf{B} and \mathbf{A} are then related as

$$\mathbf{A} = -\mathbf{B}^T, \quad (8)$$

(the upper index t denotes transposition). This relation corresponds to $k_{21} = -k_{12}$ or $h_{21} = -h_{12}$ that holds between hybrid parameters of a reciprocal two-port.

2. Evaluation of Network Functions

In the following text we shall denote the indexes of input nodes of the network as a and b and those of the output port as c and d as it is shown in Fig. 1. That implies that

$$U_{in} = U_a - U_b \quad \text{and} \quad U_{out} = U_c - U_d. \quad (9)$$

(In some special cases, of course, may be for instance $b \equiv d$).

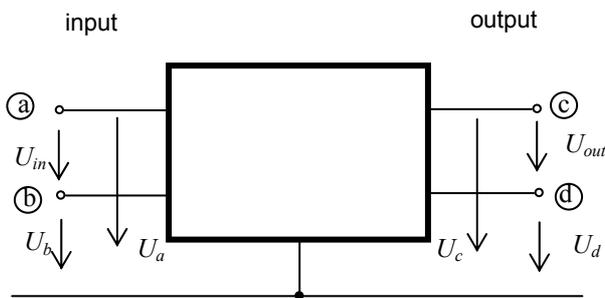


Fig. 1 Indexing the input and output terminals and voltages

When calculating transfer functions K_u or Y_t we suppose that the input is supplied from an ideal voltage source so that its current (oriented in the same direction as the voltage, Fig. 2) must be chosen as an additional current for the vector \mathbf{x} . Its index will be denoted as i .

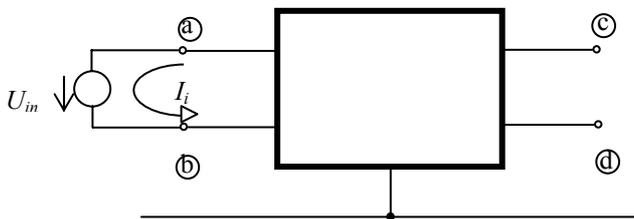


Fig. 2 Voltage source at input port

On the other hand, when calculating K_i or Y_t the output pair of terminals is short-circuited and the current must again appear in \mathbf{x} , this time with index j . The situation is shown in Fig. 3.

To obtain the required transfer function we consider a unit value for the network variable I_{in} or U_{in} in the denominator and set up the right-hand side vector accordingly. By solving the system of eqns. (5) we then obtain the vector

$$\mathbf{x} = -\mathbf{K} \setminus \mathbf{b}. \quad (10)$$

Vector \mathbf{x} contains the sought voltage $U_{out} = \mathbf{x}(a) - \mathbf{x}(b)$ or current $I_{out} = \mathbf{x}(j)$ numerically equal to the transfer function in question (symbol \setminus is inverse division in MATLAB).

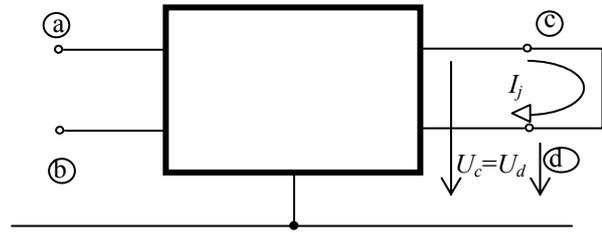


Fig. 3 Short-circuited output port

This procedure is preferable with very large networks when it is worthwhile to take advantage of \mathbf{K} -matrix sparsity. In case of relatively small networks with, say, $mn < 50$, it is best to invert the system matrix \mathbf{K} and obtain the series-parallel matrix of the system

$$\mathbf{H} = \mathbf{B}^{-1}. \quad (11)$$

Then we get all the functions easily as combination of \mathbf{H} -matrix elements:

- for the transfer impedance $Z_t = \mathbf{H}(c,a) - \mathbf{H}(c,b) - \mathbf{H}(d,a) + \mathbf{H}(d,b), \quad (12)$

- for the input impedance (when $a \equiv c$ and $b \equiv d$) $Z_{in} = \mathbf{H}(a,a) - \mathbf{H}(a,b) - \mathbf{H}(b,a) + \mathbf{H}(b,b), \quad (12a)$

- for the current transfer factor $K_i = \mathbf{H}(j,a) - \mathbf{H}(j,b), \quad (13)$

- for the voltage transfer factor $K_u = \mathbf{H}(c,i) - \mathbf{H}(d,i), \quad (14)$

- for the transfer admittance $Y_t = \mathbf{H}(j,i). \quad (15)$

Thus a single matrix inversion is necessary when it is required to obtain all possible transfer impedances (the indexes a, b, c and d may be chosen after the matrix was inverted). A single inversion is enough when seeking current transfer factors measured from any input pair of terminals a, b to the same short-circuited output j or when seeking voltage transfer factors from input i to any pair of output terminals c, d . Only the transfer admittance needs formulating and inverting the system matrix for each combination of input and output.

3. Evaluation of Absolute Sensitivities of Transfer Functions to the Variation of Element Parameters

A very important step in designing of an electronic network is evaluation of sensitivity of the network function

to the variation of parameters of all (or at least some) network elements. The study of sensitivities can reveal how important from the point of view of the required transfer characteristics is accuracy and long term stability of element parameters. This in turn indicates how expensive will be the mass production of the network. On the other hand, the sensitivities are strongly related for instance with the frequency dependence of group delay and that of the slope of modulus characteristic (so-called Bode response) in decibels per decade of frequencies.

We define 4 types of sensitivities of function F to the variation of parameter q :

- absolute sensitivity

$$S(F, q) = \frac{\partial F}{\partial q}, \quad (16)$$

- relative sensitivity

$$S_r(F, q) = \frac{\partial \ln F}{\partial \ln q} = \frac{\partial F/F}{\partial q/q} = \frac{q}{F} S(F, q), \quad (17)$$

- 2 semi-relative sensitivities

$$S_F(F, q) = \frac{\partial \ln F}{\partial q} = \frac{1}{F} S(F, q), \quad (18)$$

$$S_q(F, q) = \frac{\partial F}{\partial \ln q} = q S(F, q). \quad (19)$$

The dimensions of individual sensitivities depend on both the function F and parameter q . The relative sensitivity is dimensionless. It shows the effect of the variable parameter very clearly: the change of q by 1 ppm (part per million) will cause the change of F by S_r ppm.

Obviously, the absolute sensitivity $S(F, q)$ forms a part of all other types of sensitivities. That is why we shall analyze only the ways of obtaining $S(F, q)$ in the rest of the paper.

4. Possible Types of Variable Parameters

With no loss of generality we can suppose that the variable parameter can be admittance Y , impedance Z , conductance G , resistance R , capacitance C or inductance L of a two-terminal network element or the controlling parameter g_m , A , B or r_m of one of the 4 possible controlled sources. Parameters of more complicated elements (transistors of all types, impedance inverters or converters, current conveyors, integrated operational amplifiers and so on) can be always taken as parameters of elements in linearized equivalent schemes of these devices.

5. An Approximate Determination of the Absolute Sensitivity

The formula $S(F, q) = \partial F/\partial q$ for the absolute sensitivity can be approximated with finite increments of ΔF and Δq as

$$S(F, q) \doteq \frac{\Delta F}{\Delta q} = \frac{F(q + \Delta q/2) - F(q - \Delta q/2)}{\Delta q}. \quad (20)$$

For the determination of sensitivity it is necessary to evaluate the network function F twice: once with the value of the parameter q increased by $\Delta q/2$ and for the second time with q decreased by the same amount $\Delta q/2$. The choice of Δq determines the accuracy of computation. With large Δq the approximation of the derivative in (16) is too rough. With small values of Δq there may arise problems with numerical accuracy due to the fact that the numerator in (20) is a small difference of 2 relatively large values. The formula assumes that the dependence of F on q in the vicinity of nominal value of q is quadratic. Then the slope of tangent dF/dq in the nominal point q is the same as the slope of the secant passing two symmetrically positioned points at $q \pm \Delta q/2$ on the parabola as it is shown in Fig. 4.

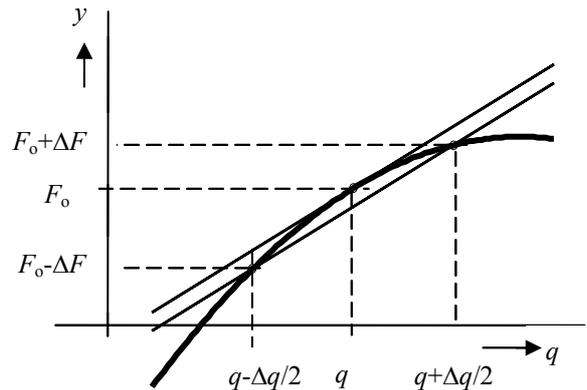


Fig. 4 Approximate calculation of the sensitivity

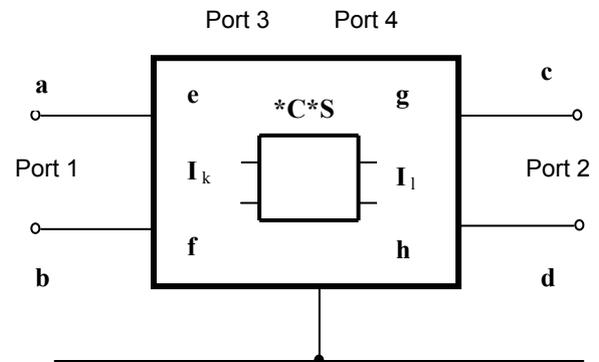


Fig. 5 Sensitivity to the parameter of a controlled source

6. An Exact Approach to the Absolute Sensitivity

For a general case, the two-port including a variable parameter element (controlled source) is shown in Fig. 5. Input of the two-port is denoted as port number 1, output as port number 2. The controlling pair of terminals e, f is 3, the output terminal g, h as port 4. The current-controlled sources (CCCS, CCVS) need an additional current i_k at port 3. The controlled voltage sources (VCVS, CCVS) need an additional current i_l at port 4.

Consider first that the source is a voltage-controlled current source VCCS characterized by transconductance g_m and the function to be simulated is $Z_T = U_{out} / I_{in}$. Under nominal conditions (nominal value of g_m) the controlling voltage

$$U_{ef} = Z_{T31} I_{in} \quad (21)$$

where Z_{T31} is transfer impedance from port 1 to port 3.

In accordance with (12)

$$Z_{T31} = \mathbf{H}(e,a) - \mathbf{H}(e,b) - \mathbf{H}(f,a) + \mathbf{H}(f,b) \quad (22)$$

Output current of the VCCS is then $g_m U_{ef}$ and output voltage of the two-port

$$U_{out} = U_c - U_d = Z_T I_{in} \quad (23)$$

with Z_T given by (12).

If, however, the transconductance g_m is slightly changed to $g_m + dg_m$ the VCCS produces additional current $dg_m U_{ef}$ directed from node g to node h and this current will cause a variation of output voltage

$$dU_{out} = dg_m U_{ef} (-Z_{T42}) \quad (24)$$

where

$$Z_{T42} = \mathbf{H}(c,g) - \mathbf{H}(c,h) - \mathbf{H}(d,g) + \mathbf{H}(d,h) \quad (25)$$

is transfer impedance from terminal pair $g-h$ (port 4) to output $c-d$ (port 2). Thus

$$\frac{dU_{out}}{dg_m} = -I_{in} Z_{T31} Z_{T42} \quad (26)$$

i.e.

$$\frac{dZ_T}{dg_m} = S(Z_T, g_m) = -Z_{T31} Z_{T42} \quad (27)$$

Consequently, **the absolute sensitivity equals the (negatively taken) product of two transfer functions** (in this special case two transfer impedances):

The first transfer function is measured from input port 1 to controlling port 3, the second one is measured from port 4 to output port 2.

A two-terminal conductance G (admittance Y) can be considered as a VCCS with ports 3 and 4 connected in parallel and transconductance $g_m = G$ ($g_m = Y$). All results remain valid, only index g is replaced by e and h by f .

Similarly, the resistance R (impedance Z) can be treated as a special case of the current-controlled voltage source CCVS. In the resultant expressions index g is again replaced by e and index h replaced by f .

For the sensitivity of the voltage transfer factor K_u to g_m we obtain by similar reasoning

$$S(K_u, g_m) = -K_{u31} Z_{T42} \quad (28)$$

Here

$$K_{u31} = U_{ef} / U_{ab} = \mathbf{H}(e,i) - \mathbf{H}(f,i) \quad (29)$$

is voltage transfer factor from port 1 to port 3 and

$$Z_{T42} = \mathbf{H}(c,g) - \mathbf{H}(c,h) - \mathbf{H}(d,g) + \mathbf{H}(d,h) \quad (30)$$

is transfer impedance from port 4 to port 2 with port 1 short circuited.

In case of other controlled sources we have to take into account additional I_k and/or I_l currents that had to be defined for these sources. The concrete form of the 2 transfer functions that appear in the absolute sensitivity formula then corresponds to the situation at the respective ports.

7. Resultant Formulae for Absolute Sensitivities

The following formulae express the individual sensitivities as the product of 2 partial transfer functions and at the same time they show how to obtain these partial functions from the hybrid matrix $\mathbf{H} = \mathbf{K}^{-1}$.

The variable parameter is the transconductance g_m of controlled source VCCS

$$S(Z_T, g_m) = -Z_{T31} \cdot Z_{T42} = -[\mathbf{H}(e,a) - \mathbf{H}(e,b) - \mathbf{H}(f,a) + \mathbf{H}(f,b)] \cdot [\mathbf{H}(c,g) - \mathbf{H}(c,h) - \mathbf{H}(d,g) + \mathbf{H}(d,h)]$$

$$S(K_i, g_m) = -Z_{T31} \cdot K_{i24} = -[\mathbf{H}(e,a) - \mathbf{H}(e,b) - \mathbf{H}(f,a) + \mathbf{H}(f,b)] \cdot [\mathbf{H}(j,g) - \mathbf{H}(j,h)]$$

$$S(K_u, g_m) = -K_{u31} \cdot Z_{T42} = -[\mathbf{H}(e,i) - \mathbf{H}(f,i)] \cdot [\mathbf{H}(c,g) - \mathbf{H}(c,h) - \mathbf{H}(d,g) + \mathbf{H}(d,h)] \quad (31 \text{ a-d})$$

$$S(Y_T, g_m) = -K_{u31} \cdot K_{i24} = -[\mathbf{H}(e,i) - \mathbf{H}(f,i)] \cdot [\mathbf{H}(j,g) - \mathbf{H}(j,h)]$$

The variable parameter is the admittance Y of a two-terminal element

$$S(Z_T, Y) = -Z_{T31} \cdot Z_{T24} = -[\mathbf{H}(e,a) - \mathbf{H}(e,b) - \mathbf{H}(f,a) + \mathbf{H}(f,b)] \cdot [\mathbf{H}(c,e) - \mathbf{H}(c,f) - \mathbf{H}(d,e) + \mathbf{H}(d,f)]$$

$$S(K_i, Y) = -Z_{T31} \cdot K_{i24} = -[\mathbf{H}(e,a) - \mathbf{H}(e,b) - \mathbf{H}(f,a) + \mathbf{H}(f,b)] \cdot [\mathbf{H}(j,e) - \mathbf{H}(j,f)]$$

$$S(K_u, Y) = -K_{u31} \cdot Z_{T24} = -[\mathbf{H}(e,i) - \mathbf{H}(f,i)] \cdot [\mathbf{H}(c,e) - \mathbf{H}(c,f) - \mathbf{H}(d,e) + \mathbf{H}(d,f)] \quad (32 \text{ a-d})$$

$$S(Y_T, Y) = -K_{u31} \cdot K_{i24} = -[\mathbf{H}(e,i) - \mathbf{H}(f,i)] \cdot [\mathbf{H}(j,e) - \mathbf{H}(j,f)]$$

The variable parameter is the amplification factor B of controlled source CCCS

$$S(Z_T, B) = -K_{i31} \cdot Z_{T24} = -[\mathbf{H}(k,a) - \mathbf{H}(k,b)] \cdot [\mathbf{H}(c,g) - \mathbf{H}(c,h) - \mathbf{H}(d,g) + \mathbf{H}(d,h)]$$

$$S(K_i, B) = -K'_{i31} \cdot K_{i24} = -[\mathbf{H}(k,a) - \mathbf{H}(k,b)] \cdot [\mathbf{H}(j,g) - \mathbf{H}(j,h)]$$

$$S(K_u, B) = -Y_{T31} \cdot Z_{T24} = -\mathbf{H}(k,i) \cdot [\mathbf{H}(c,g) - \mathbf{H}(c,h) - \mathbf{H}(d,g) + \mathbf{H}(d,h)]$$

$$S(Y_T, B) = -Y'_{T31} \cdot K_{i24} = -\mathbf{H}(k,i) \cdot [\mathbf{H}(j,g) - \mathbf{H}(j,h)] \quad (33 \text{ a-d})$$

The variable parameter is the amplification factor A of controlled source VCVS

$$S(Z_T, A) = -Z_{T31} \cdot K_{u24} = -[\mathbf{H}(e,a) - \mathbf{H}(e,b) - \mathbf{H}(f,a) + \mathbf{H}(f,b)] \cdot [\mathbf{H}(c,l) - \mathbf{H}(d,l)]$$

$$S(K_i, i) = -Z_{T31} \cdot Y_{T24} = -[\mathbf{H}(e,a) - \mathbf{H}(e,b) - \mathbf{H}(f,a) + \mathbf{H}(f,b)] \cdot \mathbf{H}(j,l)$$

$$S(K_u, A) = -K_{u31} \cdot K'_{u24} = -[\mathbf{H}(e,i) - \mathbf{H}(f,i)] \cdot [\mathbf{H}(c,l) - \mathbf{H}(d,l)]$$

$$S(Y_T, A) = -K'_{u31} \cdot Y'_{T24} = -[\mathbf{H}(e,i) - \mathbf{H}(f,i)] \cdot \mathbf{H}(j,l) \quad (34 \text{ a-d})$$

The variable parameter is the transresistance r_m of controlled source CCVS

$$S(Z_T, r_m) = -K_{i31} \cdot K_{u24} = -[\mathbf{H}(k,a) - \mathbf{H}(k,b)] \cdot [\mathbf{H}(c,l) - \mathbf{H}(d,l)]$$

$$S(K_i, r_m) = -K'_{i31} \cdot Y_{T24} = -[\mathbf{H}(k,a) - \mathbf{H}(k,b)] \cdot \mathbf{H}(j,l)$$

$$S(K_u, r_m) = -Y_{T31} \cdot K'_{u24} = -\mathbf{H}(k,i) \cdot [\mathbf{H}(c,l) - \mathbf{H}(d,l)]$$

$$S(Y_T, r_m) = -Y'_{T31} \cdot Y'_{T24} = -\mathbf{H}(k,i) \cdot \mathbf{H}(j,l) \quad (35 \text{ a-d})$$

The variable parameter is the impedance Z of a two-terminal element

$$S(Z_T, Z) = -K_{i31} \cdot K_{u24} = -[\mathbf{H}(k,a) - \mathbf{H}(k,b)] \cdot [\mathbf{H}(c,k) - \mathbf{H}(d,k)]$$

$$S(K_i, Z) = -K'_{i31} \cdot Y_{T24} = -[\mathbf{H}(k,a) - \mathbf{H}(k,b)] \cdot \mathbf{H}(j,k)$$

$$S(K_u, Z) = -Y_{T31} \cdot K'_{u24} = -\mathbf{H}(k,i) \cdot [\mathbf{H}(c,k) - \mathbf{H}(d,k)]$$

$$S(Y_T, Z) = -Y'_{T31} \cdot Y'_{T24} = -\mathbf{H}(k,i) \cdot \mathbf{H}(j,k) \quad (36 \text{ a-d})$$

8. Examples

The network with 7 independent nodes in Fig. 6 consists of 10 conductances and 1 controlled source. The reference node is given serial number 8. The resultant number

of unknown network variables depends on the number of currents that must be added to the vector of 7 node voltages in accordance with the type of the required transfer function and type of the controlled source.

Values of the conductances were selected quite randomly by a rand function as $G_1 = 2.0907$, $G_2 = 3.7982$, $G_3 = 7.8333$, $G_4 = 6.8085$, $G_5 = 4.6110$, $G_6 = 5.6783$, $G_7 = 7.9421$, $G_8 = 5.9183 \cdot 10^{-2}$, $G_9 = 6.0287$, $G_{10} = 5.0269 \cdot 10^{-1}$. Various combinations of simulated transfer functions with different controlled sources were then calculated in order to check the formulae for sensitivities. In all the cases the respective parameter $q = g_m$, $q = A$, $q = B$ or $q = r_m$ of the controlled source was supposed to be variable. The approximate method of finite differences used always $\Delta q/2 = 0.005$. Obtained values were then compared with those obtained by exact formulae based on inverse matrix $\mathbf{H} = \mathbf{K}^{-1}$.

Some of the obtained results are presented in Tab. 1. In the column "Sensitivity" there is also shown the used nominal value of controlled source parameter. The next column contains values of the transfer function corresponding to this nominal parameter value. The last two columns show the results of sensitivity calculations.

9. Conclusion

The modified node voltage method makes it possible to calculate both the network functions and their sensitivities to variation of element parameters. Especially in case of relatively modest sizes of networks when the system matrix can be inverted economically all network functions and sensitivities can be obtained in a very effective manner.

This procedure is not so advantageous for large networks with sparse system matrices. The sparsity is lost during matrix inversion. In such a case it is better to evaluate the necessary partial transfer functions for sensitivity calculations by repeatedly solving the equations with different right-hand sides. In some cases it may be advantageous to use the concept of adjoint network [3] that is inter-reciprocal with the original network and has system hybrid matrix

$$\mathbf{K}_a = \begin{bmatrix} \mathbf{Y}_a & \mathbf{B}_a \\ \mathbf{A}_a & \mathbf{Z}_a \end{bmatrix} \quad (37)$$

The sub-matrices are related to those in (6):

$$\mathbf{Y}_a = \mathbf{Y}^T, \quad \mathbf{B}_a = -\mathbf{A}^T, \quad \mathbf{A}_a = -\mathbf{B}^T, \quad \mathbf{Z}_a = \mathbf{Z}^T. \quad (38)$$

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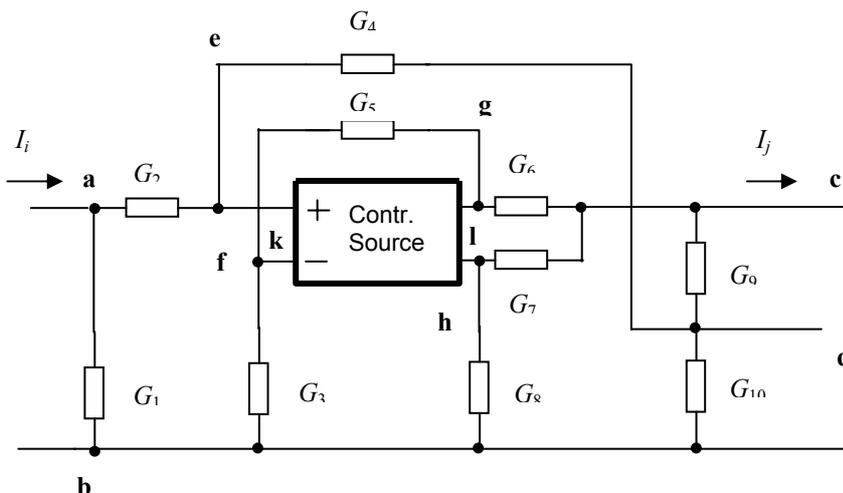


Fig. 6 A test network for calculation of sensitivities

	Sensitivity	Nominal function value	Finite differences	Exact formulae
1	$S(Z_T, g_m), g_m=2$	-0.03272906540957	+0.005989705	+0.00598970320381
2	$S(K_u, g_m), g_m=2$	-0.09811743550769	+0.021468232	+0.02146823026047
3	$S(K_i, B), B=5$	-1.58059824313269	+0.5852127	+0.58520803699831
4	$S(Z_T, A), A=12$	-0.08163522241084	-0.000630109	-0.0006301092035
5	$S(Y_T, r_m), r_m=4$	-2.13221693334707	-0.087782774	-0.0877827725542
6	$S(K_u, r_m), r_m=4$	-0.33106862885119	-0.017665146	-0.0176651291582

Tab. 1 Results of selected examples

