Optimized State Model of Piecewise-Linear Dynamical Systems

Jiří POSPÍŠIL[1], *Zdeněk KOLKA*[1], *Stanislav HANUS*[1], *Václav MICHÁLEK*[1], *Jaromír BRZOBOHATÝ*[2]

[1] Dept. of Radio Electronics, Brno University of Technology, Purkyňova 118, 612 00 Brno, Czech Republic
[2] Dept. of Microelectronics, Brno University of Technology, Údolní 53, 602 00 Brno, Czech Republic

pospisil@feec.vutbr.cz

Abstract. *The conditions for an optimized design of the second dynamical system having low eigenvalue sensitivities are directly derived. Their more general form is in accordance with the previous results obtained by using linear topological conjugacy.*

Keywords

Dynamical systems, piecewise-linear systems, sensitivity, optimization.

1. Introduction

Recently published new state models of piecewise-linear (PWL) dynamical systems of Class *C* can be used as prototypes for their circuit realization. For this purpose their eigenvalue sensitivities have been minimized using linear topological conjugacy [1], first for linear systems and then applied to PWL systems. By detailed analysis of the second order systems the new optimization conditions giving minimum sum of relative eigenvalue sensitivity squares with respect to the change of the individual state matrix parameters have been derived [2]. Applying them separately to the cases of real and complex conjugate eigenvalues, two different forms of the optimized state matrix are obtained [3]. In this contribution the generalized form of directly derived optimization conditions is introduced. It evidently includes both previous forms [3] as two special cases.

2. Basic Principles

Consider second-order linear dynamical system described by general state matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}\,, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} . \tag{1}$$

Its characteristic equation

$$\det(s\mathbf{1}-\mathbf{A}) = s^2 - (a_{11}+a_{22}) + (a_{11}a_{22}-a_{12}a_{21}) = \\ = (s-\lambda_1)(s-\lambda_2) = 0 \tag{2}$$

has the roots given by

$$a_{11}+a_{22} = \lambda_1 + \lambda_2\,, \quad a_{11}a_{22}-a_{12}a_{21} = \lambda_1\lambda_2\,. \tag{3a,b}$$

The relative eigenvalue sensitivities with respect to the individual state matrix parameters are

$$S_r(\lambda_k, a_{11}) = \frac{a_{11}(\lambda_k - a_{22})}{\lambda_k\left[2\lambda_k - (a_{11}+a_{22})\right]}\,, \tag{4a}$$

$$S_r(\lambda_k, a_{22}) = \frac{a_{22}(\lambda_k - a_{11})}{\lambda_k\left[2\lambda_k - (a_{11}+a_{22})\right]}\,, \tag{4b}$$

$$S_r(\lambda_k, a_{12}) = S_r(\lambda_k, a_{21}) = \\ = \frac{a_{12}a_{21}}{\lambda_k\left[2\lambda_k - (a_{11}+a_{22})\right]}\,, \quad k = 1,2 \tag{4cd}$$

They evidently satisfy to the basic sensitivity invariant condition [2], i.e. $\Sigma\, S_r\,(\lambda_k, a_{ij}) = 1$.

3. Generalized Conditions

Utilizing the basic formulas (3a,b), the sum of the relative sensitivity squares is obtained in the form

$$\sum S_r^{\,2}(\lambda_k, a_{ij}) \;=\; \frac{(a_{11}^2 + a_{22}^2)\,\lambda_k^2 - 2a_{11}a_{22}(a_{11}+a_{22})\lambda_k + 2(a_{11}^2a_{22}^2 + a_{12}^2a_{21}^2)}{\left[\lambda_k(a_{11}+a_{22}) - 2(a_{11}a_{22} - a_{12}a_{21})\right]^2}\,,$$

$$k = 1, 2. \tag{5}$$

In accordance with the chosen sensitivity measure [2], the first derivatives of the expression (5) with respect to the individual state matrix parameters are calculated and then the corresponding generalized conditions for their zero values can be determined, i.e.

$$\frac{\partial}{\partial a_{11}} \sum S_r^{\,2}(\lambda_k, a_{ij}) = 0\,, \tag{6a}$$

$$\frac{\partial}{\partial a_{22}} \sum S_r^{\,2}(\lambda_k, a_{ij}) = 0\,, \tag{6b}$$

$$\frac{\partial}{\partial a_{12}} \sum S_r^{\,2}(\lambda_k, a_{ij}) = 0 \quad , \tag{6c}$$

$$\frac{\partial}{\partial a_{21}} \sum S_r^{\,2}(\lambda_k, a_{ij}) = 0 \quad . \tag{6d}$$

Starting from the (6a,b), the general optimized parameters a_{11} and a_{22} are obtained as

$$a_{11} = a_{22} = \frac{1}{2}\left(\lambda_1 + \lambda_2\right) \tag{7}$$

that corresponds to the basic design formula (3a), while the conditions (6c,d) finally entails the common generalized formula for the product of parameters a_{12} and a_{21}

$$a_{12}\, a_{21} = \frac{1}{4}\left(\lambda_1 - \lambda_2\right)^2 \tag{8}$$

that evidently corresponds to the basic design formula (3b).

4. Detailed Results for the Individual Special Cases

In the second-order systems three basic cases of two eigenvalues can exist:

4.1 Two Different Real Eigenvalues

Here $\lambda_1 \neq \lambda_2$ and the corresponding state matrix can be rewritten into the form [3]

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} \lambda_1 + \lambda_2 & (\lambda_1 - \lambda_2)K \\ (\lambda_1 - \lambda_2)K^{-1} & \lambda_1 + \lambda_2 \end{bmatrix} \tag{9}$$

and the coefficient K is a free parameter utilizable in optimized design procedure also for PWL dynamical systems of Class C. The corresponding eigenvalue sensitivities and the resultant optimized sensitivity measures are summarized in Table 1.

4.2 Two Identical Real Eigenvalues

Then $\lambda_1 = \lambda_2 = \lambda$ and the corresponding state matrix becomes to the simplest Jordan form

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad . \tag{10}$$

The simplified eigenvalue sensitivities and sensitivity measures are:

$$S_r(\lambda_1, a_{11}) = S_r(\lambda_1, a_{22}) = S_r(\lambda_2, a_{11}) = S_r(\lambda_2, a_{22}) = \frac{1}{2}$$

$$S_r(\lambda_1, a_{12}) = S_r(\lambda_1, a_{21}) = S_r(\lambda_2, a_{12}) = S_r(\lambda_2, a_{21}) = 0 \ ,$$

$$\sum S_r^2(\lambda_1, a_{ij}) = \sum S_r^2(\lambda_2, a_{ij}) = \frac{1}{2} \ . \tag{11a-c}$$

λ_k	λ_1	λ_2
$S_r(\lambda_k, a_{11}) =$ $= S_r(\lambda_k, a_{22})$	$\frac{1}{4}\left(1 + \frac{\lambda_2}{\lambda_1}\right)$	$\frac{1}{4}\left(1 + \frac{\lambda_1}{\lambda_2}\right)$
$S_r(\lambda_k, a_{11}) =$ $= S_r(\lambda_k, a_{22})$	$\frac{1}{4}\left(1 - \frac{\lambda_2}{\lambda_1}\right)$	$\frac{1}{4}\left(1 - \frac{\lambda_1}{\lambda_2}\right)$
$\sum S_r^2(\lambda_k, a_{ij})$	$\frac{1}{4}\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^2\right]$	$\frac{1}{4}\left[1 + \left(\frac{\lambda_1}{\lambda_2}\right)^2\right]$

Tab. 1. The individual eigenvalue sensitivities and the optimized sensitivity measures.

4.3 Two Complex Conjugate Eigenvalues

In this case $\lambda_1 = \lambda' + j\lambda''$, $\lambda_1 = \lambda' - j\lambda''$. Substituting them into general conditions (7) and (8), we obtain the special simplified forms

$$a_{11} = a_{22} = \lambda' , \qquad a_{12}\, a_{21} = -\left(\lambda''\right)^2 \tag{12}$$

and the corresponding state matrix can be rewritten into the so-called complex decomposed form, including the optimization coefficient K [3]

$$\mathbf{A} = \begin{bmatrix} \lambda' & \lambda'' K \\ -\lambda'' K^{-1} & \lambda' \end{bmatrix} . \tag{13}$$

In this case all the sensitivity functions are obtained in the complex form and the same functions, expressed separately for the eigenvalues real and imaginary parts, can easily be derived. Then the optimum sensitivity measures are

$$\sum S_r^2(\lambda', a_{ij}) = \sum S_r^2(\lambda'', a_{ij}) = \frac{1}{2} . \tag{14}$$

5. Statistical Evaluation

The system sensitivity properties could be easily evaluated with the Monte-Carlo method. Let us suppose for simplicity the elements of the system matrix **A** are statistically independent with the equal variance. By means of repeated generation of random parameters it is possible to obtain an estimation of probability density function of resulting eigenvalues that determine qualitative behavior of the studied system. Fig. 1 shows eigenvalue scatter plots for the complex case. The optimum realization (Fig. 1a) is compared with the equivalent eigenvalue form (Fig. 1b) with the matrix

$$\mathbf{A} = \begin{array}{|c|c|} \hline p_1 & -1 \\ \hline p_2 & 0 \\ \hline \end{array} \ , \tag{15}$$

where $p_1 = \lambda_1 + \lambda_2$, $p_2 = \lambda_1 \lambda_2$,. Both forms had the same statistical properties of **A**.

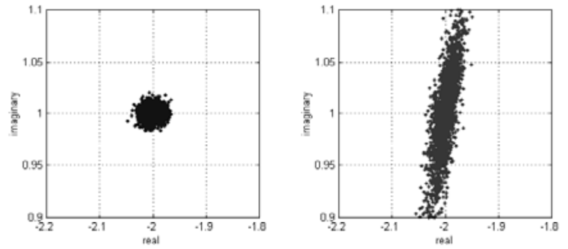

Fig. 1. Comparison of sensitivity properties for system with complex eigenvalues $\lambda_{1,2} = -2 \pm j$..

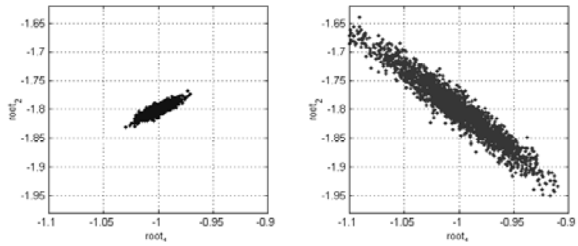

Fig. 2. Comparison of sensitivity properties for system with real eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1.8$.

Fig. 2 shows the same comparison for the real case. It is evident; the optimum form minimizes the influence of variation of circuit parameters to the system dynamics.

6. Conclusion

The resultant general equations (7) and (8) include a degree of freedom for the optimized model design for all types of eigenvalues. (In PWL systems of Class C it is represented by additional optimization coefficient [3].) In case of higher-order systems such a direct design procedure is not possible and the block-decomposed state matrix can be used [4].

Acknowledgements

This research is partially supported by the Grant Agency of the Czech Republic, Grant projects No. 102/02/1312/A and No. 102/01/0229. It represents the part of the Research Program of the Czech Ministry of Education – CEZ: J22/98: 262200011.

References

[1] WU, C.W., CHUA, L.O. On Linear Topological Conjugacy of Lur'e Systems. *IEEE Trans. Circ. Syst. - I: Fundamentals*..., 1996, 43(2), p. 158-161.

[2] KOLKA, Z. Synthesis of Optimized Piecewise-Linear System Using Similarity Transformation–Part I: Basic Principles. *Radioengineering*, 2001, vol. 10, no.3, p.5-7.

[3] POSPÍŠIL, J., KOLKA, Z., HORSKÁ, J. Synthesis of Optimized Piecewise-Linear System Using Similarity Transformation – Part II: Second-Order Systems. *Radioengineering*, 01, vol. 10, no. 3, p. 8-10.

[4] POSPÍŠIL, J., KOLKA, Z., HANUS, S., BRZOBOHATÝ, J. Synthesis of Optimized Piecewise-Linear System Using Similarity Transformation – Part III: Higher-Order Systems. *Radioengineering*, 2002, vol. 11, no. 1, p. 4-6.

About Authors...

Jiří POSPÍŠIL was born in Brno, Czechoslovakia, in 1939. M.Sc. and Ph.D. (equiv. degrees): 1963 and 1973, respectively; DSc. (equiv. degree.): 1988, all in el. engg, TU Brno, Czechoslovakia. 1964: Assist. Prof., Military Acad. of Brno, Dept of El. Engg; 1970-1972: Visit. Prof., Military Tech. College, Cairo, Egypt; since 1974: TU Brno, Dept of Radioelectronics; 1980: Assoc. Prof.; 1989: Prof.; Research and pedag. interest: Circuits and Systems Theory, *PWL* Dynam. Networks, Dynam. Systems Modeling. IEEE: M.-1992, S.M.- 1995.

Zdeněk KOLKA was born in Brno, Czechoslovakia, in 1969. He received the M.S. (92) and Ph.D. (97) degrees in electrical engineering, both from the Faculty of Electrical Engineering and Computer Science, Brno Univ. of Technology. At present he is an Assistant Professor at the Institute of Radio Electronics. He is interested in PWL modeling, circuit simulation, and nonlinear dynamical systems.

Jaromír BRZOBOHATÝ was born in Brno, Czechoslovakia, in 1935. M.Sc. and Ph.D. (equiv. degrees): 1960 and 1980, respectively, both in el. engg, TU Brno, Czechoslovakia. 1960-1963: researcher in Metra Blansko; 1963: Assoc. Prof.; 1987: Prof.; 1985-1989: Dean of Faculty of El. Engg; 1980-1993: Head of Dept of Microelectronics. Research and pedag. interest: Circuits Theory, Microelectronics, *PWL* Dynam. Netw. IEEE: M.-1988, S.M.-1990.

Stanislav HANUS was born in Brno, Czechoslovakia, in 1950. He received the Ing. (M.Sc.) and CSc. (Ph.D.) degrees from the Brno University of Technology. He is Associated Professor at the Institute of Radio Electronics, FEEC, BUT Brno. His research is concentrated on Circuit theory and Wireless and Mobile Communications.