Compaction Filter as an Optimum Solution for Multirate Subband Coder of Cyclostationary Signals

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Abstract. *A consistent theory of optimum subband coding of zero mean wide-sense cyclostationary signals with N-periodic statistics is presented in this paper. Blocked polyphase representation of the analysis and synthesis filter banks is introduced as an effective way of multirate subband coder description. Optimum energy compaction using Nyquist-M process is presented as a solution for maximizing the coding gain of the coder. In two definitions and four theorems the author proves that Nyquist-M filters fulfill necessary and sufficient conditions imposed on subband signals. Results from Matlab simulations are presented to support theoretical conclusions***.**

Keywords

Coding gain, decimation, filter bank, polyphase representation, power spectral density matrix, multirate system, subband coding, compaction filter.

1. Introduction

A structure of the Multirate Subband Coder for Wide-sense Cyclostationary (WSCS) signals, that is to be treated hereinafter, is depicted in Fig. 1. Sets of filters $H_i(k, z^{-1})$ and $F_i(k, z^{-1})$, represent the Analysis Bank and the Synthesis Bank respectively. The index k indicates time varying nature of these blocks, since each of them in fact consists of a sequence of N Linear Time-Invariant (LTI) filters, where N is assigned to the periodicity of cyclostationarity of a Linear Periodically Time-Varying (LPTV) structure. For the sake of simplicity we assume that period of time variation of the structure in Fig. 1 is the same as that of input signal. Blocks to the left of the analysis bank are M-fold decimators that discard all but every M-th sample. Blocks to the right of the synthesis bank are M-fold interpolators that raise the sampling rate by a factor of M, by inserting (M-1) zero samples between two consecutive samples of an incoming stream. Block Q_i stands for i-th A/D converter, communication channel and i-th D/A converters together.

Cyclostationarity of the input signal is apprehended in wide sense for the purpose of this work, assuming N periodicity of the second order statistics.

One of the crucial quantities used to evaluate efficiency of subband coder is a coding gain, defined by (1.1) as a ratio between the output distortion variance of a simple PCM coder and that of the subband coder

$$G_{SBC} = \frac{MN\sigma_x^2}{\sum_{k=0}^{N-1}\left(\prod_{i=0}^{M-1}\sigma_{vi}^2(k)\right)^{1/M}} \quad . \tag{1.1}$$

Here σ_x^2 represents the input signal variance and $\sigma_{vi}^2(k)$ represents variance of i-th subband signal in time instant k, see Fig. 2. For an optimum subband coder, the coding gain shall be maximized for given statistical parameters of the input sequence $x(n)$, under optimum bit allocation scheme

$$2^{-2b_i(k)}\sigma_{vi}^2(k) = 2^{-2b_{i+1}(k)}\sigma_{v(i+1)}^2(k) \quad , \tag{1.2}$$

where $b_i(k)$ stands for number of bits, allocated to the A/D converter in i-th subband channel, at the time instant k within the period of cyclostationarity N.

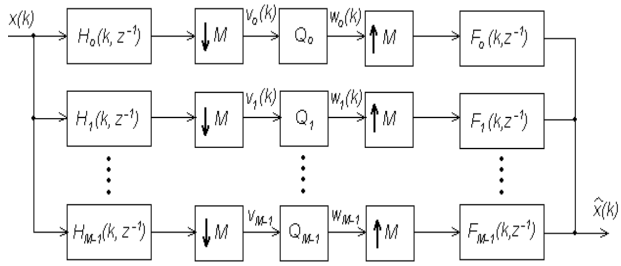

Fig. 1. Multirate subband coder for WSCS signals with maximally decimated filter bank.

LTI equivalent of the multirate subband coder is depicted in Fig. 2. Blocked polyphase matrices $\tilde{E}(z^{-1})$ and $\tilde{R}(z^{-1})$ represent analogy to the analysis and synthesis bank. Vectors $\overline{X}(k)$, $\overline{V}(k)$ and $\overline{Q}(k)$ are Wide-sense Stationary (WSS) representations of the input signal, subband signals and quantizing noise, respectively.

Among many alternatives to express analysis blocked polyphase matrix $\tilde{E}(z^{-1})$, the one in (1.3) introduced in [7], uses matrix transfer functions $\tilde{H}_{ij}(z^{-1})$. The $N\times N$ matrix transfer functions $\tilde{H}_{i0}(z^{-1})$, $\tilde{H}_{i1}(z^{-1})$, …, $\tilde{H}_{i(M-1)}(z^{-1})$, respectively relate the N-fold blocked 1-st, 2-nd, …, M-th samples within a period of M samples, to N-fold blocked M-th samples of the output of the analysis filter $H_i(k, z^{-1})$.

$$\tilde{E}\left(z^{-1}\right)=\begin{bmatrix}\tilde{H}_{00}\left(z^{-1}\right) & \cdots & \tilde{H}_{0(M-1)}\left(z^{-1}\right)\\ \vdots & \ddots & \vdots\\ \tilde{H}_{(M-1)0}\left(z^{-1}\right) & \cdots & \tilde{H}_{(M-1)(M-1)}\left(z^{-1}\right)\end{bmatrix} \tag{1.3}$$

$$\tilde{H}_{mn}(z^{-1})=\sum_{k=-\infty}^{\infty}\tilde{h}_{mn}(k)z^{-k} \tag{1.4}$$

Z-transform (1.4) couples the transfer functions to respective $N\times N$ impulse response matrix. Due to the periodic nature of the original structure [7], each pq-element of the matrix represents an LTI system,

$$\begin{aligned}&\left[\tilde{h}_{mn}(k,l)\right]_{pq}=\\ &h\left[M(Nk-p)-m, M(Nl-q)-n\right]=\\ &\left[\tilde{h}_{mn}(k-l)\right]_{pq}\end{aligned} \quad . \tag{1.5}$$

To design the analysis and synthesis filter bank, i.e. to calculate the elements of polyphase matrices $E^{\sim}\exp(-j\omega)$ and $R^{\sim}\exp(-j\omega)$ is far from being a simple task. Necessary and sufficient conditions stated for Power Spectrum Matrix (PSD) of subband signals $v_i(k)$ in order to reach optimality are reviewed in the following theorem. For proof see [9].

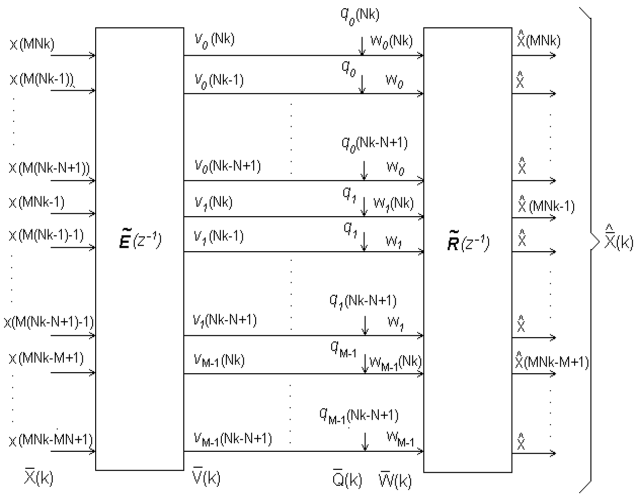

Fig. 2. Blocked polyphase representation of the filter banks.

Theorem 1.1: The optimum performance of the multirate subband coder with uniform, maximally decimated filterbank for WSCS signals, i.e. the maximum coding gain (1.1) is attained, if and only if both of the following hold:

1. The subband signals $v_i(k)$ are totally decorrelated for all k, i.e. the blocked PSD matrix $S_V(\omega)$ is diagonal.
2. The diagonal elements of $S_V(\omega)$ obey a specific magnitude ordering at each ω, although possibly through a different frequency invariant permutation, than that indicated by the canonic ordering.

Designer's task is to tailor out the filter banks to meet the above conditions.

2. Optimum Energy Compaction

The main intention of this section of the paper is to show formally, that the analysis filters yielded by the optimum solution, which itself calls for a canonical ordering of the subband variances, as required by Theorem 1.1, is in fact an optimum compaction filter for WSCS input signal.

2.1 Nyquist-M filter

Definition and a list of fundamental properties of Nyquist-M filters (or M-th band filters) used for WSS signals are presented in [8].

Consider a polyphase decomposition of $H(z^{-1})$; see Fig.1, with omitted index k. Suppose the 0-th polyphase component $E_0(z^{-1})$ is a constant c, making

$$H(z^{-1})=c+z^{-1}E_1(z^M)+\ldots+z^{-(M-1)}E_{M-1}(z^M)\ . \tag{2.1}$$

Then the output

$$Y(z^{-1})=cX(z^M)+\sum_{l=1}^{M-1}z^{-l}E_l(z^M)X(z^M)\ . \tag{2.2}$$

Expression (2.2) implies that $y(Mn)=cx(n)$ in the time domain. In practical applications one can scale the filter such that $c=1$. Thus, even though the interpolation filter inserts new samples, the existing samples in the input sequence $x(n)$ are communicated to the output without distortion.

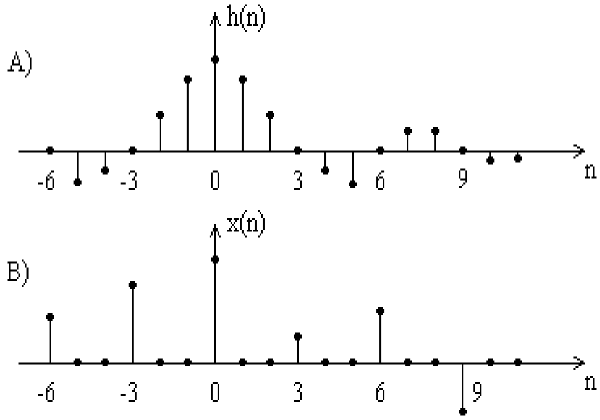

Fig. 3. Impulse response of the Nyquist-M filter (A) and example of the input sequence (B).

An impulse response having the above property satisfies condition

$$h(Mn)=\begin{cases}c,\ldots n=0\\ 0,\ldots\text{otherwise}\end{cases}\ . \tag{2.3}$$

In other words, $h(n)$ has periodic zero-crossings separated by M samples, with exception of $h(0)=c$. See Fig. 3-A, which demonstrates this property for $M = 3$. In Fig. 3-B, a typical appearance of the input sequence for this filter $x(n)$ is depicted. After convolving $x(n)$ with the impulse response $h(n)$, nonzero samples of $x(n)$ are unaffected, except for a scaling factor c.

In frequency domain (Z-domain), the Nyquist-M property is manifested as well. If $H(z^{-1})$ satisfies (2.1), it can be shown that

$$\sum_{k=0}^{M-1} H(z^{-1}W_M^k) = Mc \quad , \tag{2.4}$$

where core of the DFT $W_M = \exp(-j2\pi / M)$. As depicted in Fig. 4, the frequency response of $H(z^{-1} W_M^k)$ is the shifted version $H\{\exp[-j(\omega+2k\pi / M)]\}$ of $H[\exp(-j\omega)]$. Finally, we can conclude that all M uniformly shifted versions of $H[\exp(-j\omega)]$ add up to an allpass filter. Covering complete relative frequency band $(0, 2\pi)$ corresponds to the Power Complementary (PC) property of the filter bank.

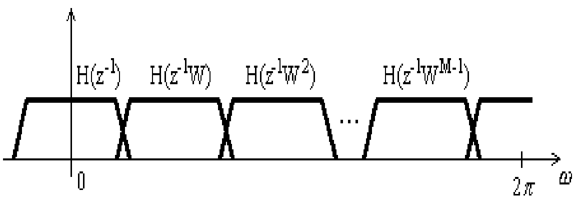

Fig. 4. Images of an impulse response of the Nyquist-M filter.

2.2 Nyquist-M Process

To deal with optimum energy compaction solution for the subband coder considered in this paper, one must first define N-periodic optimum compaction of an N-periodic WSCS process. Definition of optimum compaction of WSS signals described in [4], involves an LTI filter $H(z^{-1})$ for which $H(z^{-1}) H^*(z)$ is Nyquist-M. For an LPTV system $H(k, z^{-1})$ let's employ the *adjoined filter* $H^a(k, z^{-1})$, firstly defined for Nyquist-2 filter by Schwartz particularly for 2-channel subband coder in [7], whose impulse response $h^a(k, l)$ relates to the impulse response $h(k, l)$ of $H(k, z^{-1})$

$$h^a(k,l) = h^*(l,k) \quad . \tag{2.5}$$

It can be observed [8] that the adjoined of an LTI system with transfer function $H(z^{-1})$, has transfer function $H^*(z)$. Thus analogy of system with transfer function $H(z^{-1}) H^*(z)$ in the LTV (Linear Time-Varying) case is the LTV system $H(k, z^{-1}) H^a(k, z^{-1})$. Following definition constitutes LTV Nyquist-M filter.

Definition 2.1: LTV Nyquist-M Filter

Consider the arrangement in Fig. 5-A with $H(k, z^{-1})$ LTV filter with impulse response $h(k, l)$. Then $H(k, z^{-1})$ is Nyquist-M if for all integers n and m, following equality holds

$$h(Mn, Mm) = c\delta(n-m) \quad , \tag{2.6}$$

where δ denotes Kronecker delta.

Clearly, should $H(k, z^{-1})$ be an LTI filter $H(z^{-1})$, (2.6) reduces to the definition given for LTI Nyquist-M filters (2.3). However, at the first sight it may appear that the following formula represents more direct analogy to the LTI case. For all integers m, n

$$h(Mn+m, m) = c\delta(n) \quad . \tag{2.7}$$

However, there are at least two reasons to favour weaker requirement (2.6). The first one, and more pertinent for this reasoning, is related to the following consequence of the Nyquist-M property in LTI case. Referring again to Fig.5-A with $H(k, z^{-1})$ being LTI $H(z^{-1})$ with type one polyphase components $E_0(z)$, $E_1(z)$, …, $E_{M-1}(z)$. Then, $H(z^{-1}) H^*(z)$ is Nyquist-M if and only if for all ω

$$\left[E_0(e^{j\omega}), E_1(e^{j\omega}), \ldots, E_{M-1}(e^{j\omega})\right] \begin{bmatrix} E_0^*(e^{-j\omega}) \\ E_1^*(e^{-j\omega}) \\ \vdots \\ E_{M-1}^*(e^{-j\omega}) \end{bmatrix} = 1 \ . \tag{2.8}$$

One can easily observe, that $E_0(z)$, $E_1(z)$, …, $E_{M-1}(z)$ respectively represent the LTI systems relating the 0-th, 1-st, …, $(M-1)$-th samples within a period of M samples, to the M-th samples of $s(k)$. It is exactly this fact that is used to link the optimum compaction process to an optimum solution for $H_0(z^{-1})$ in the WSS case.

Recall from definitions in section 1, that $\tilde{H}_{00}(z^{-1})$, $\tilde{H}_{01}(z^{-1})$, …, $\tilde{H}_{0(M-1)}(z^{-1})$ respectively represent an LTI systems relating the blocked 0-th, 1-st, …, $(M-1)$-th samples within a period of M samples, to the blocked M-th samples of $s(k)$, when $H(k, z^{-1})$ is WSCS with period N. Thus, the comparable result for WSCS case would be as follows: for N-periodic $H(k, z^{-1})$, the product $H(k, z^{-1}) H^a(k, z^{-1})$ is Nyquist-M if and only if for all ω,

$$\left[\tilde{H}_{00}(e^{-j\omega}), \ldots, \tilde{H}_{0(M-1)}(e^{-j\omega})\right] \begin{bmatrix} [\tilde{H}_{00}(e^{-j\omega})]^H \\ [\tilde{H}_{01}(e^{-j\omega})]^H \\ \vdots \\ [\tilde{H}_{0(M-1)}(e^{-j\omega})]^H \end{bmatrix} = \bar{I} \ . \tag{2.9}$$

In general, values of $h(k,l)$ playing a part in (2.9) are only those that correspond to the M-th values of k within a period of M samples. On the other hand (2.7) also affects 0-th, 1-st, …, $(M-1)$-th values of k in $h(k,l)$. As general consequence, (2.7) implies (2.9), but isn't equivalent to (2.9).

The second reason for preferring (2.6) goes beyond the application of Nyquist-M filters in this work. As already indicated in subsection 2.1 for WSS case, a key advantage of LTI Nyquist-M filters dwells in their usefulness for M-fold interpolation process, see Fig. 5-B.

Then, an LTI $H(z^{-1})$ is Nyquist-M if and only if for all n and $x(n)$, $y(Mn)=x(n)$. Obviously, only the M-th samples of the input and output of $H(z^{-1})$ in Fig. 5-B are pertinent to this requirement. Thus, condition (2.7) demands more than needed. Theorem 2.1 below shows that (2.6) is indeed equivalent to the above interpolation requirement.

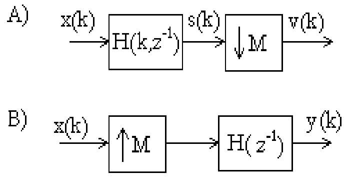

Fig. 5. Analysis filter cascaded with decimator (A) interpolator cascaded with interpolation filter (B).

Theorem 2.1: In Fig. 5-B, suppose $H(z^{-1})$ is LTV filter $H(k, z^{-1})$. Then for all k and $x(k)$,

$$y(Mk) = x(k) \tag{2.10}$$

if and only if $H(k, z^{-1})$ is Nyquist-M.

Proof: Clearly, if (2.10) is expressed as the requirement that for all $x(k)$ and k

$$y(Mk) = \sum_{l=-\infty}^{\infty} h(Mk, Ml)x(l) = x(k) \ , \tag{2.11}$$

it is satisfied by using (2.6) with $c = 1$.

It is also known that in LTI case, $H(z^{-1})$ is Nyquist-M if and only if $E_0(z) = c$, possibly conveniently with $c = 1$. Then, the following Theorem 2.2 proves a comparable result for the N-periodic LPTV case.

Theorem 2.2: In Fig. 5-B, suppose $H(z^{-1})$ is LPTV filter $H(k, z^{-1})$ with period N. Consider $\tilde{H}_{00}(z^{-1})$ defined in (1.3), (1.4). Then $H(k, z^{-1})$ is Nyquist-M, if and only if

$$\tilde{H}_{00}(z^{-1}) = \bar{I} \ . \tag{2.12}$$

Proof: Because of $[\tilde{h}_{mn}(k, l)]_{pq} = [\tilde{h}_{mn}(k - l)]_{pq}$ with $l = 0$, (2.12) is equivalent to the requirement, that for all k and

$$\begin{aligned} &0 \le p, \quad q \le N-1 \ , \\ &h\left[M(Nk-p), -Mq\right] = \\ &= \delta(k)\,\delta(p-q) = \quad , \\ &= \delta(Nk-p+q) \end{aligned}$$

where the second equality follows from $0 \le p, q \le N$–1. Because of N-periodicity of $h(k,l)$ this is equivalent to (2.6). Detailed proof for 2-channel filter bank is available in [7].

2.3 Optimum Energy Compaction with Nyquist-M Filter

In the following section, reader's attention shall be turned to the Nyquist-M property of the blocked submatrices $\tilde{H}_{00}(z^{-1}), \tilde{H}_{01}(z^{-1}), \ldots, \tilde{H}_{0(M-1)}(z^{-1})$, respectively.

Theorem 2.3: Consider an N-periodic analysis filter $H(k, z^{-1})$, its adjoined filter $H^a(k, z^{-1})$ and blocked submatrices $\tilde{H}_{mn}(z^{-1})$, defined in (1.3) and (1.4) for $m, n = 0, 1$ to M-1. Then $H(k, z^{-1})\, H^a(k, z^{-1})$ is Nyquist-M if and only if (2.9) holds.

Proof: Consider a structure in Fig. 6 and define for $i = 0, 1$ to M-1,

$$r_i(k) = r(Mk - i) \ . \tag{2.13}$$

Define the N-fold blocked version of $r_i(k)$ as

$$\tilde{r}_i(k) = \left[r(MNk-i), \ldots r(M(Nk-N+1)-i)\right]^{\mathrm{T}} \tag{2.14}$$

and

$$\tilde{r}(k) = \left[\tilde{r}_0^{\mathrm{T}}(k), \tilde{r}_1^{\mathrm{T}}(k), \ldots, \tilde{r}_{M-1}^{\mathrm{T}}(k)\right]^{\mathrm{T}} \ . \tag{2.15}$$

Then the blocked polyphase LTI system relating $\tilde{s}(k) = [\tilde{s}_0^{\mathrm{T}}(k), \tilde{s}_1^{\mathrm{T}}(k), \ldots, \tilde{s}_{M-1}^{\mathrm{T}}(k)]^{\mathrm{T}}$ $\{\tilde{s}_i^{\mathrm{T}}(k)$, is defined in the same manner as (2.14)$\}$ to $\tilde{r}(k)$, has transfer function

$$\tilde{E}^a\left(z^{-1}\right) = \begin{bmatrix} \tilde{H}_{00}^{\mathrm{H}}(z) & \cdots & \tilde{H}_{(M-1)0}^{\mathrm{H}}(z) \\ \vdots & \ddots & \vdots \\ \tilde{H}_{0(M-1)}^{\mathrm{H}}(z) & \cdots & \tilde{H}_{(M-1)(M-1)}^{\mathrm{H}}(z) \end{bmatrix} . \tag{2.16}$$

Thus, the system relating $\tilde{x}_0(k)$ to $\tilde{r}_0(k)$ has transfer function

$$\begin{aligned} \left[\tilde{E}^a\left(z^{-1}\right)\tilde{E}\left(z^{-1}\right)\right]_{00} &= \tilde{H}_{00}\left(z^{-1}\right)\tilde{H}_{00}^{\mathrm{H}}(z) + \ldots \\ &\ldots + \tilde{H}_{(M-1)0}\left(z^{-1}\right)\tilde{H}_{(M-1)0}^{\mathrm{H}}(z) \end{aligned}$$

The result then follows from Theorem 2.2, expression (2.12), establishing analogy to the LTI case mentioned earlier.

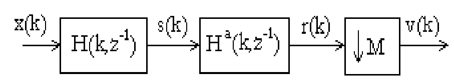

Fig. 6. Concatenation of the analysis filter, its adjoined counterpart and the M-fold decimator.

Definition 2.2: Optimum Compaction Process

Consider Fig. 5-A, with $x(k)$ WSCS with period N, and $H(k, z^{-1})$ LPTV with period N. Then, $H(k, z^{-1})$ is an optimum compaction filter for $x(k)$, if subject to being Nyquist-M and for some index set $\{k_0, k_1, \ldots, k_{N-1}\} = \{0, 1, \ldots, N\text{-}1\}$ it simultaneously maximizes partial sum of variances of $v(k)$:

$$\sum_{i=0}^{l} \sigma_v^2(k_i) \ , \tag{2.17}$$

for all $0 \le l \le N$–1, $0 \le v \le M$–1.

It can be observed, that the above definition targets to accommodate the fact that $v(k)$ is WSCS with period N. Consequently, N variance values in total have to be considered. One can call the optimum compaction filter a *canonical filter*, if in (2.17) $k_i = i$. It shall be noted however, that even canonical filter is not unique. This is consistent with properties of compaction filters for WSS processes [4].

The main result of this paper follows.

Theorem 2.4: Recall Fig. 1 with $H_i(k, z^{-1})$ N-periodic and $x(k)$ WSCS with period N. Then the $H_0(k, z^{-1})$ provided by the solution to the Theorem 1.1 is an N-periodic optimum compaction filter for $x(k)$.

Proof: Consider any N-periodic canonical optimum compaction filter $H(k, z^{-1})$ of signal $x(k)$. Consider the $N \times MN$ matrix

$$\tilde{H}_0(e^{-j\omega}) = \left[\tilde{H}_{00}(e^{-j\omega}), \tilde{H}_{01}(e^{-j\omega}), .. \tilde{H}_{0(M-1)}(e^{-j\omega})\right] \tag{2.18}$$

with $H^{\sim}_{mn}(z^{-1})$ for $m, n = 0, 1, \ldots, M-1$ defined in (1.3) and (1.4). We can observe that by Definition 2.2, $H(k, z^{-1})$ is Nyquist-M filter. Hence, referring to Theorem 2.3, condition (2.9) holds. For $H(k, z^{-1})$ being a canonical optimum compaction filter, the sum of partial variances of subband signals

$$\sum_{i=0}^{l} \int_0^{2\pi} \left[\tilde{H}_0\left(e^{-j\omega}\right) S_{\overline{X}}(\omega)\, \tilde{H}_0^{\mathrm{H}}\left(e^{-j\omega}\right) \right]_{ii} d\omega \ , \tag{2.19}$$

has to be maximized for all $0 \le l \le N-1$. The expression inside brackets represents a unitary transform of positive semi-definite Hermitian symmetric matrix $S_x(\omega)$. Due to properties of Hermitian symmetric matrices and under (2.9),

$$\sum_{i=0}^{l} \left[\tilde{H}_0\left(e^{-j\omega}\right) S_{\overline{X}}(\omega)\, \tilde{H}_0^{\mathrm{H}}\left(e^{-j\omega}\right) \right]_{ii} \tag{2.20}$$

is maximized for all $0 \le l \le N-1$, if and only if

$$\left[\tilde{H}_0\left(e^{-j\omega}\right) S_{\overline{X}}(\omega)\, \tilde{H}_0^{\mathrm{H}}\left(e^{-j\omega}\right)\right]_{ii} = \lambda_i(\omega) \ , \tag{2.21}$$

where $\lambda_i(\omega)$ are the eigenvalues of S_X. Notation ii refers to diagonal elements of PSD matrix.

When arguing in a similar way as in [9], the implication is such that partial sums in (2.19) are simultaneously maximized if and only if

$$\tilde{H}_0\left(e^{-j\omega}\right) S_{\overline{X}}(\omega)\, \tilde{H}_0^{\mathrm{H}}\left(e^{-j\omega}\right) = \mathrm{diag}\left\{\lambda_0(\omega), .., \lambda_{N-1}(\omega)\right\} \tag{2.22}$$

where $\lambda_n(\omega) \ge \lambda_{n+1}(\omega)$ for almost all ω. These are conditions from Theorem 1.1, met by $H_0(k, z^{-1})$. Hence the result.

3. Simulation Model

In the previous section a consistent derivation was provided, showing that optimum compaction filter is one possible solution to the maximization of the coding gain of the multirate subband coder with maximally decimated filter banks.

To verify efficiency of subband coding as far as practical expectations on coding gain values are concerned, a Matlab simulation model of the multirate subband coder has been created, adopting the results derived above in this paper and in [9].

A topology of the two-channel filter bank used for simulation is depicted in Fig. 6. LPTV analysis FIR filters $H_i(z^{-1}, N)$ and synthesis filters $F_i(z^{-1}, N)$, represented by their respective impulse responses h_{i1}, h_{i2} and f_{i1}, f_{i2}, are pertinent to be used for WSCS signal with $N = 2$. In other words, a WSCS input waveform is processed by the filters, which alternate between impulse responses $h_{i1}$and h_{i2}, as well as between f_{i1} and f_{i2}. While action of decimators and interpolators has already been described in this work, quantities q_{i1}, q_{i2} substitute for a quantizing noise in the Matlab model. The average power of the quantizing noise depends on the statistic of particular subband signal v_i and number of bits allocated to the ADC, as governed by (1.2). Due to the Periodically Dynamic Bit Allocation (PDBA), simulated source of quantizing noise also alternates between values $q_{i1}$and q_{i2}.

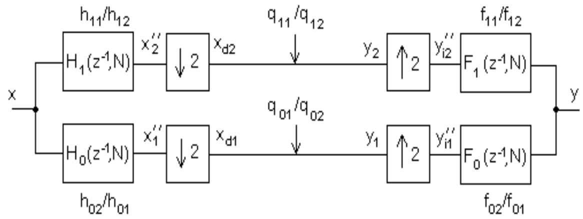

Fig. 6. Two-channel maximally decimated filter bank used for simulation.

Since it is not a trivial task to obtain a real cyclostationary signal with period $N = 2$ to serve as an input for the subband coder, two artificial signals were synthesized instead. The first input sequence

$$x_1(2i-1) = 1 \cdot \sin\left(\frac{2\pi}{8} i\right),$$
$$x_1(2i) \ = 0.5 \cdot \sin\left(\frac{6\pi}{8} i\right), \tag{3.1}$$

obviously $E[x(k)] = m_x(k) = m_x(k+2)$ holds. The second test input sequence

$$x_2(i) = 1 \cdot \sin\left(\frac{\pi}{8} i\right) + 0.25 \cdot \sin\left(\frac{7\pi}{8} i\right) \tag{3.2a}$$

for $0 \le (i \bmod 2P) \le P$ and

$$x_2(i) = 0.75 \cdot \sin\left(\frac{\pi}{8} i\right) + 0.5 \cdot \sin\left(\frac{7\pi}{8} i\right) \tag{3.2b}$$

for $P \le (i \bmod 2P) \le 2P$, for some large number P.

Discrete spectra of both input test trains are depicted in Fig.7 - A, B with windowing effects neglected. Both the above-presented trains are no doubt purely deterministic waveforms, hence being stationary in nature. But for the purpose of subsequent simulations, these interlaced sinusoidal waveforms will serve as a trivial representation of WSCS signal, with individual sinusoid representing a WSS element within a period of cyclostationarity $N = 2$.

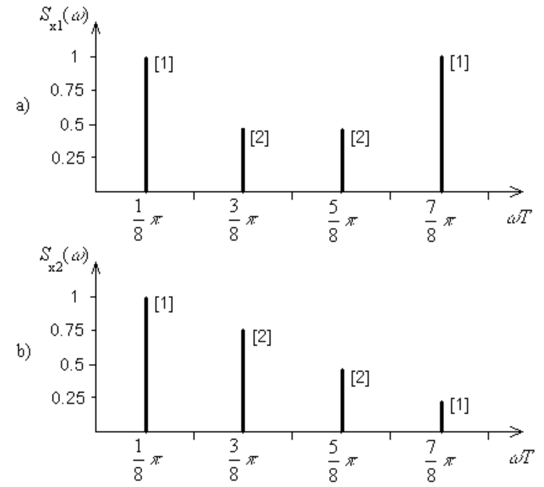

Fig. 7. Discrete spectra of input test waveforms, (A) defined by formula (3.1) and (B) formula (3.2).

The core element in the Matlab model used to demonstrate subband coder functionalities is a power symmetric Quadrature Mirror Filter-bank (QMF), designed for the simplest case of two-channel ($M = 2$) subband coder with period of cyclostationarity $N = 2$. Power symmetric QMF itself represents special class of alias free digital systems with low complexity and reasonably low amplitude distortion (3.3)

$$T(z) = \frac{1}{2}\,[H_0(z)\,H_1(-z) - H_1(z)\,H_0(-z)\,]\,. \tag{3.3}$$

For detailed derivation of how to compensate for the aliasing and amplitude distortion in the QMF structure, the reader shall kindly refer to [8]. Since the QMF will be employed as an element in more complex structure, only most important results are reviewed below.

Assuming that low-pass $H_0(z)$ used in QMF is power symmetric, it can be seen, that if the high-pass $H_1(z)$ is designed so that

$$H_1(z) = -z^{-N}\tilde{H}_0(-z)\ , \tag{3.4}$$

for some odd N, then (3.3) is reduced into $T(z) = 0.5\,z^{-N}$, i.e. to the perfect reconstruction system. To achieve a realizable system, filters $H_0(z)$, $H_1(z)$ have to be FIR (to avoid instability of their paraconjugate counterparts used in power symmetry definition). The synthesis filters are then given by [8]

$$F_0(z) = z^{-N}\tilde{H}_0(z)\ ,\ F_1(z) = z^{-N}\tilde{H}_1(z)\,. \tag{3.5}$$

All the above formulae for analysis filter $H_1(z)$ and synthesis filters can be respectively rewritten in time domain as

$$h_1(k) = (-1)^k h_0^*(N-k)\ , \tag{3.6a}$$

$$f_0(k) = h_0^*(N-k)\ , \tag{3.6b}$$

$$f_1(k) = (-1)^k h_0(N-k)\ , \tag{3.6c}$$

reducing the design of the QMF into the design of causal and power symmetric low-pass $H_0(z)$.

Totally four filter designs were tested by the author of this paper, differing in stop-band attenuation and relative transition bandwidth, as reviewed in Table 2. The design procedure itself comprises a calculation of zero-phase half-band filter $H(z) = H^{\sim}_0(z)\,H_0(z)$ of an even order $2K$, for some odd K, using Sparks-McClellan algorithm from Matlab. The low-pass filter $H_0(z)$ of order K, was then extracted from $H(z)$ by spectral factorization. Resulting normalized distortion function of QMF (3.3), exhibits ripple as low as $0.5{\cdot}10^{-3}$ dB for least stringent design specification and $3{\cdot}10^{-3}$ dB for most stringent one.

design number	min.stopband attenuation	transition bandwidth	half-band filter order	stopband peak ripp.
1	30 dB	0.1 pi	74	$4.54 \cdot 10^{-4}$
2	30 dB	0.3 pi	26	$2.35 \cdot 10^{-4}$
3	20 dB	0.1 pi	50	$3.61 \cdot 10^{-3}$
4	20 dB	0.3 pi	18	$1.95 \cdot 10^{-3}$

Tab. 1. Specifications of the filters used for simulation.

Basic topology of the simulation model for WSCS-like signal is depicted in Fig. 8. The outward QMF, which consists of the filters $H_0(z^{-1})$, $H_1(z^{-1})$, $F_0(z^{-1})$, $F_1(z^{-1})$ and works with input signal x and output signal y, cuts the total frequency band by half, allocating low-frequency and high-frequency components to the subband signals x'_1 and x'_2, respectively. Indeed, using two non-overlapping filters in the analysis bank brings decorrelation to the subband signals. Since for the chosen test inputs, as defined by (3.1) through (3.2b), the energy of the train of pulses of sinusoidal waveforms is well concentrated far from π/2, the decorrelation is almost perfect.

The purpose of the imbedded QMF structure, having its outputs connected to switches, dwells in the decorrelation of the frequency components that originate from samples (blocks of samples in case of x_2) arriving to the input of the WSCS subband coder in different time instants, within a period of cyclostationarity. By using synchronous switching of the time-multiplexed components of the input signal, the simulation structure effectively creates four non-overlapping filters of equal bandwidth, that separate frequency components of the input signal, hence decorrelating them in the sense required by Theorem 1.1, condition 1.

As for the amplitudes (variances), test input signals are adjusted in such way, that largest component in spectrum is paired with the smallest and the second largest component is paired with the second smallest. Since the spectrum of the signal has non-zero values in a narrow area around discrete spectral lines, the subband signals approximately fulfill the second rule of Theorem 1.1.

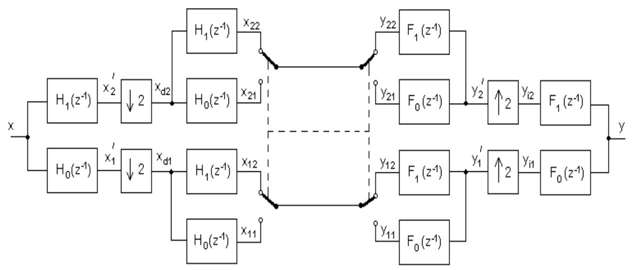

Fig. 8. Topology of the simulation model with tree-structured QMF.

To evaluate the coding gain reached by the application of subband coding on a particular input signal, the appropriate quantizing noise model had to be implemented into Matlab code. In the simulation model, a random noise with uniform probability density function is generated by `rand()` function from Matlab tool-kit. Values of the noise are distributed within amplitude interval <-0.5; 0.5>, the variance is adjusted via suitable multiplicative constant. Such noise model is believed to represent well a real A/D converter. The same random sequence is used to derive values of the quantizing noise for both the subband signals v_i and those of reference quantizing noise of a PCM coder.

Results gained from Matlab simulation model, that carries out processing of WSCS signals by a subband coder with maximally decimated filter banks, are listed in Tab. 2, respectively for test input trains x_2 and x_1.

For all four designs examined and for input signal x_2, the coding gain exceeds 10 dB and varies a little with changing the filter order. This insensitivity to the change of the filter order shall be contributed to the narrowband character of the spectra of test input sequence, which doesn't contain much energy in the vicinity of subband borders. Three simulations were performed for each design and averages were calculated.

WSCS filter bank simulation results signal X2			
design no.	**quant. noise**	**ref. qnt. noise**	**cod. gain [dB]**
1	0.223	2.44	10.34
3	0.228	2.44	10.31
2	0.226	2.43	10.32
4	0.233	2.44	10.19

WSCS filter bank simulation results signal X1			
design no.	**quant. noise**	**ref. qnt. noise**	**cod. gain [dB]**
1	1.53	6.57	6.34
3	1.59	6.46	6.09
2	1.62	6.55	6.06
4	1.63	6.45	5.98

Tab. 2. Coding gain results for test signals X2 and X1.

Due to the symmetry of spectra of x_1 with respect to $\pi/2$, coming from the nature of sum of two interpolated sinusoids, the coding cannot benefit from pairing (ordering) of the subband signals according to the second rule of Theorem 1.1. In fact both lines in each pair formed within the period of cyclostationarity $N = 2$ have the same energy. Hence the coding gain of 6.1 dB shall be fully attributed to the decorrelation of subband signals. This result allows the reader to build a specific feeling for judging individually the importance of the two rules specified in Theorem 1.1.

3.1 Conclusion on Simulation Results

The results show coding gain of **10.3 dB** for input signal x_2, which has canonical decay of PSD towards higher frequencies, and **6.1 dB** for input signal x_1, having PSD symmetric with respect to relative frequency $\pi/2$. Comparative results for WSS subband coder show coding gain of **4.2 dB** for signal x_2. Hence ad hoc expectation of the coding gain of two-channel subband coder for WSCS signals then falls within the range 6 – 12 dB, indicating one or two bit savings with respect to standard PCM coder.

Such results are believed to justify complexity of presented theory and hopefully will encourage further investigations in the area of subband coding.

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