Nonlinear Resistor with Polynomial AV Characteristics and Its Application in Chaotic Oscillator

Jiří PETRŽELA, Václav POSPÍŠIL

Dept. of Radio Electronics, Brno University of Technology, Purkyňova 118, 612 00 Brno, Czech Republic

xpetrz07@stud.feec.vutbr.cz, xpospi13@stud.feec.vutbr.cz

Abstract. This paper shows the realization of two terminal devices with an arbitrary polynomial nonlinearity up to the fifth order. The proposed design procedure is completely systematic using minimum of components. The very heart of our conception is four-channel four-quadrant analog multiplier MLT04.

The implementation of synthesized nonlinear resistor as a general nonlinearity in chaotic oscillator is also presented and experimentally verified.

Keywords
Analog multiplier, chaos, polynomial nonlinearity, oscillator.

1. Introduction

The study of dynamical systems with nonlinear elements has been a topic of increasing interest for last two decades. This concern is especially obvious in the field of chaotic oscillators. However, there is no commercially available product with well-defined smooth nonlinearity. To eliminate this disadvantage several approaches were developed for synthesizing piecewise linear AV or transfer characteristics. These procedures are mostly based on controlled sources as VCVS, VCCS or negative impedance converters [2].

The behavior of a dynamical system is given by its equations and initial conditions. In some rare cases of the third-order dynamical systems, nor periodic, nor stable solutions occur when time goes to infinite. For these systems the chaotic attractor is observed. The possibility of chaos presence is about 0.5% and it grows with the dimension of studied system. The major property of chaotic system is in extreme sensitivity to tiny changes of initial state and long complex unpredictable behavior. We concentrate only on implicitly given differential equation of the type \( \dddot{x} + a \dddot{x} + \dot{x} + x - f(x) = 0 \), which exhibits chaotic behavior for \( a=0.6 \) and some piecewise linear, polynomial and even goniometrical functions \( f(x) \). Integrating of each term of this equation uncovers that it is a damped harmonic oscillator driven by a nonlinear memory term involving the integral of \( f(x) \). Systems of the form \( \dddot{x} = F(\dot{x}, \dddot{x}, x) \) have been called jerk equations (time derivative of acceleration).

2. Design of a Nonlinear Resistor

It has been proved that analog multipliers can be used as a basic building block in resistors with monotonic AV characteristics. Using of the off-the-shelf device AD633 leads to a compact network structure with high sweep of input voltage ±11V and possible extension to higher-order polynomial. Transfer function of these IC is

\[
W = K (X_1 - Y_1)(X_2 - Y_2) + Z
\]

where \( K=0.1 \) denotes a scaling factor of an output voltage and \( X_1, Y_1, Z \) are independent input voltages.

The need of many IC since one package contains only one multiplier is the main disadvantage. For example, the fifth-order polynomial with all nonzero coefficients can be realized by five packages. This circuit is suggested in [4]. We can eliminate this drawback by using MLT04. This analog multiplier has maximal range of input voltage ±2.5V, low power supply ±5V and good temperature stability. Total linearity error is essential for our purpose and is typically only ±2% of the full scale. The primary static errors in an analog multiplier are input and output offset voltages. The input sources of errors can be eliminated by using the optional trim circuit. MLT04 includes four multipliers; each of them has its individual ground and realizes a simple transfer function

\[
W_i = K X_i Y_i, \quad i = 1, 2, 3, 4.
\]

The driving point characteristics is given as the 5th order polynomial of the form

\[
i = f(u) = a_5 u^5 + a_4 u^4 + a_3 u^3 + a_2 u^2 + a_1 u + a_0.
\]

The elementary operation in this manner is squaring. A compensation of internally set constant \( K=0.4±0.02 \) isn’t allowed, although this factor is cumulated in higher-order coefficients of the polynomial.

Let’s specify the positions of switches as \( S_i=±1 \). Then individual coefficients of (3) can be easily derived from the schematic in Fig. 9 as follows.
\[ a_0 = \frac{5}{R_5} \left( \frac{2 R_a}{R_a} - 1 \right), \quad a_1 = \sum_{i=1}^{2} \frac{1}{R_i} - \frac{1}{R_6}, \quad a_2 = -S_1 \frac{K}{R_1}, \]
\[ a_3 = -\prod_{i=1}^{2} S_i \frac{K^2}{R_2}, \quad a_4 = -\prod_{i=1}^{3} S_i \frac{K^3}{R_3}, \quad a_5 = -\prod_{i=1}^{4} S_i \frac{K^4}{R_4}. \]

where \( R_a \) is a part of a variable resistor connected to \(-5V\), which is supposed to be linear due to the average value of coefficient \( a_0 \) close to zero. A voltage follower is here employed as an isolation buffer. The input resistance of the inverter (consisting of the operational amplifier U1 and two affiliated resistors) has a negligible effect on final characteristics. Some examples of nonlinearities are demonstrated in Fig. 1. Our concretization is not convenient for approximation of goniometrical functions beyond one period. Let’s recall the Taylor’s power row formulae for sine function

\[
\sin x = \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} (-1)^{n-1} x^{2n-1}.
\] (5)

We can do this after using many multiplication cells.

3. Universal Chaotic Oscillator

In this section we discuss the circuitry implementation of mentioned chaotic oscillator. It is well known that 3\textsuperscript{rd} order differential equation can be rewritten into the matrix form, i.e. into the set of 1\textsuperscript{st} order differential equations. Our system being recast has the form similar to dynamical systems of class C [6], in detail

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \\ q_3 - q_2 \\ q_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ q_1 & q_2 & q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ q_3 - q_2 \\ q_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ x \end{pmatrix} \cdot f(x).
\] (6)

Due to the dissipativity of (6) for bounded chaotic attractor any continuous function \( f(x) \) must have at least one zero at \( x = \bar{x} \). The stability of this fixed point is determined by the solution of the equilibrium problem \( \det (\lambda E - J) = 0 \), where \( E \) is unity matrix and \( J \) denotes Jacobi matrix

\[
\lambda^3 - q_1 \lambda^2 + q_2 \lambda - q_3 - \frac{\partial f(x)}{\partial x} = 0
\] (7)
evaluated at \( x = \bar{x} \). This point is then locally stable for (8a) and undergoes a Hopf bifurcation (complex conjugate eigenvalues cross the imaginary axis) if (8b)

\[ q_1 q_2 < q_3 + \frac{\partial f(x)}{\partial x} < 0 \quad q_1 q_2 - q_3 = \frac{\partial f(x)}{\partial x}. \]

The transformation that preserves \( \partial f(x)/\partial x \) and the graph of \( f(x) \) only affects the size of the attractors. As follows from (6) our dynamical system deals with equivalent eigenvalue parameters \( q_1 = -0.6, q_2 = 1 \) and \( q_3 = -1 \). In spite of its algebraical simplicity another modification can be applied. Huge numerical searches reveal chaotic cases with \( q_1 = 0 \), namely the dual volume conserving cases with remaining parameters \( q_2 = 4, q_3 = -1 \). From the chaos point of view the principal quantity is Lyapunov exponent, which measures the divergence of two neighboring trajectories. The Poincaré-Bendixon’s theorem implies the necessity of three state variables so there must be just three Lyapunov exponents. According to the natural definition of chaos one exponent must be positive and one equals to zero. The last one must be negative and its absolute value must be the largest as the attractor is loosing its energy. The variety of feedback functions, eigenvalues as well as base-e largest Lyapunov exponent (marked as LE) are documented in Tab. 1. Note that at least one equilibrium point is of the type of unstable spiral.

For a double scroll attractor only two sets of coefficients in (3) are important, namely \( a_3 = a_4 = a_2 = a_0 = 0 \) and \( \text{sign}(a_3) \neq \text{sign}(a_1 + q_3) \). It is

![Fig. 1. Measured nonlinearities specified in Tab. 1.](image)
because of restriction on existence of three equilibrium points

$$\begin{align*}
\bar{x}_1 &= (0 \ 0 \ 0)^T \\
\bar{x}_{2,3} &= \left( \pm \sqrt{-\frac{a_1-q_1}{a_3}} \ 0 \ 0 \right)^T.
\end{align*}$$

(9)

The dynamical systems producing a single folded band structure of attractors have quadratic nonlinearity $a_5 = a_4 = a_3 = 0$, $a_1 = -q_1$, $\text{sign}(a_2) \neq \text{sign}(a_0)$ and two equilibrium points

$$\begin{align*}
\bar{x}_{1,2} &= \left( \pm \sqrt{-\frac{a_0}{a_2}} \ 0 \ 0 \right)^T.
\end{align*}$$

(10)

Individual trajectories were also numerically simulated by means of MathCAD 2000 using the 4th order Runge-Kutta iteration method with time interval $(0, 300)$, iteration step 0.03 and initial conditions $(0.1, 0, 0)$. 

### Table 1. Topology of attractors.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>LE</th>
<th>Eigenvalues at equilibria</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm 0.4 x^2 \mp 0.2$</td>
<td>0.073</td>
<td>$-0.464 \pm 0.068 \pm i 0.965$</td>
</tr>
<tr>
<td>$1.6 x^3 - 0.6 x$</td>
<td>0.103</td>
<td>$-1.067 \pm 0.254 \pm i 1.202$</td>
</tr>
<tr>
<td>$-0.177 x^3 + 1.85 x$</td>
<td>0.103</td>
<td>$0.531 \pm 0.566 \pm i 1.132$</td>
</tr>
<tr>
<td>$x^n \quad n = 2, 3, 4, 5$</td>
<td>0.002</td>
<td>$-0.836 \pm 0.118 \pm i 1.088$</td>
</tr>
<tr>
<td>$x^2 + x - 0.29$</td>
<td>0.018</td>
<td>$-0.872 \pm 0.136 \pm i 1.104$</td>
</tr>
</tbody>
</table>

The linear part of (6) can be intuitively constructed by a cascade connection of elementary inverting integrators. Then terms in the last equation are realized by feedback and summing amplifier. However, this design procedure is definitely not the simplest. Another way is demonstrated in Fig. 10. We can employ the so-called Antoniou’s general impedance converter as high order admittance

$$Y(s) = LC_1 C_2 s^3 + R_g C_1 C_2 s^2.$$  

(11)

Further details are presented in [1]. State variable $x$ is then chosen to be the voltage on the linear capacitor $C_3$ and state variable $y$ is a current that flows through it. If both functional blocks are connected in parallel the proposed state equation is

$$f(u) = C_3 \frac{du}{dt} + R_g C_1 C_2 \frac{d^3 u}{dt^3} + LC_1 C_2 \frac{d^3 u}{dt^3}.$$  

(12)

By this direct approach we obtain a circuit with unrealistic parameter values, namely resistors in Ohms, capacitors in Farads and inductance in Henries. Although it is possible to replace an inductor with its floating synthetic equivalent, such network has a drawback in two GIC requirements as it is illustrated in Fig. 11. Anyway, we are forced to introduce current $i_c = 1000$ and time $t_c = 10000$ rescaling. While doing it, resistors $R$ alike as capacitors $C$ and inductor $L$ change their values

$$\begin{align*}
R_{res} &= r_c R_s = 10^5 R_s \\
L_{res} &= r_c r_s^{-1} L_s = 0.1 L_s \\
C_{res} &= r_c^{-1} r_s^{-1} C_s = 10^{-7} C_s.
\end{align*}$$

(13)

Some of circuit parameters can be fixed

$$R_f = 1 \Omega, \quad C_1 = C_2 = C_3 = 100 nF.$$
To preserve integration speed and chaotic behavior the rest of the GIC parameters should be computed

\[ R_g = 900 \, \Omega , \quad L = 300 \, mH . \]

On the contrary to \( R_a = 100 \, k\Omega \) for variable resistors \( R_i = 1 \, M\Omega \) a logarithmic \( R_a \) characteristics is expected. Classical LM1458 can be employed to provide standard mathematical operations. It is cheap and commercially available device which contains two individual operational amplifiers. Sufficient performance in frequency domain guarantees trustworthy state trajectories with quick and repeated bursts.

PSpice 9.0 verification with Final Time=200ms and Step Ceiling=20\( \mu \)s of single folded band structure is shown in Fig. 5. This is the case with omitted INIC (S→1) and other values are

\[ R_0 = 1 \, M\Omega \quad R_1 = 1 \, M\Omega \quad R_2 = 1 \, M\Omega \quad R_3 = 1 \, M\Omega \]
\[ R_4 = 1 \, M\Omega \quad R_5 = 1 \, M\Omega \quad R_6 = 1 \, M\Omega \quad R_a = 80 \, k\Omega \]

Directly from (4), it is not hard to establish the value of \( R_b \) according to completely eliminating of term \( a_i = 0 \)

\[
R_b = \frac{\prod_{i=0}^{5} R_i}{\sum_{j=0}^{5} \delta_j} \quad \delta_j = \prod_{j=0}^{5} R_j . \tag{14}
\]

The state trajectory for the double scroll attractor is in Fig. 6. In this case, the switch S is in position 1 and the values of the resistors are

\[ R_0 = 1 \, M\Omega \quad R_1 = 1 \, M\Omega \quad R_2 = 900 \, \Omega \quad R_3 = 1 \, M\Omega \]
\[ R_4 = 1 \, M\Omega \quad R_5 = 1 \, M\Omega \quad R_6 = 1.5 \, k\Omega \quad R_a = 50 \, k\Omega . \]

Without INIC we obtain the following listing

\[ R_0 = 1 \, M\Omega \quad R_1 = 1 \, M\Omega \quad R_2 = 900 \, \Omega \quad R_3 = 1 \, M\Omega \]
\[ R_4 = 1 \, M\Omega \quad R_5 = 1 \, M\Omega \quad R_6 = 600 \, \Omega \quad R_a = 50 \, k\Omega . \]

A torus is simulated in Fig. 7 with the switch in S→1 and the parameters of the resistors

\[ R_0 = 1 \, M\Omega \quad R_1 = 1 \, M\Omega \quad R_2 = 900 \, \Omega \quad R_3 = 1 \, M\Omega \]
\[ R_4 = 1 \, M\Omega \quad R_5 = 1 \, M\Omega \quad R_6 = 4.1 \, k\Omega \quad R_a = 50 \, k\Omega . \]

Other quasiperiodic system can be displayed when the switch is in position 1 and the values of the resistors

\[ R_0 = 7 \, k\Omega \quad R_1 = 400 \, \Omega \quad R_2 = 1 \, M\Omega \quad R_3 = 1 \, M\Omega \]
\[ R_4 = 1 \, M\Omega \quad R_5 = 8 \, k\Omega \quad R_6 = 500 \, \Omega \quad R_a = 40 \, k\Omega . \]

The good final agreement should be recognized by comparing individual trajectories simulated by using PSpice 9.0 with numerical results in Fig. 2, 3 and 4.
Fourier spectrum of chaotic signals is continuous and limited. The biggest distinguishable frequency component is about 5 kHz, so it is well into the audio range and many nonlinear phenomena can be heard. Bifurcation sequence goes through period doubling and can be easily traced. The evolution of the double scroll attractor, that means the process of merging two single scroll structures together, has been also watched (see Fig. 8). It stands to reason that for solving the dynamical system with $q_1 = 0$ the lossy inductor must be interchanged with an ideal one. Frequency spectrum is wider, spread roughly from zero to 100 kHz. Kirchhoff’s laws give the equivalent inductance

$$u_{\phi_s} - u_{\phi_s} = i_s sCR^2 \quad \rightarrow \quad L_{eq} = 300 \text{ mH}$$

(15)
4. Conclusion

Although the limitations are evident, the circuit implementation of universal chaotic oscillator with smooth nonlinearity was developed. This circuit works in hybrid voltage/current mode. Soft tuning of circuit parameters allows us to observe different structures of chaotic attractors. Physical realization would provide us with interesting tool for laboratory experiments. Realization could be made with using only four IC. INIC is not necessary for the correct function of our circuit, but it is well suited for quickly swapping between consumption and distribution regime of preadjusted nonlinearity.

Undoubtedly, chaotic oscillators will play the significant role in future communication techniques. Such system that produces chaotic trajectories with several shapes can be very useful, mostly in coders, modulators and broadband signal generators. In these applications it is supposed that nonlinear resistor simplifies into the form with two or three multiplication cells. For dynamical systems (6) we can deduce the mathematical model of synchronization based on modern control theory, as it is briefly discussed in [16].

Acknowledgements

The research described in the paper was financially supported by the Czech Grant Agency under the grant projects No. 102/03/H105 and No. 102/04/0469. The authors would like to thank the reviewers for many useful advices about the concept of this paper and the Analog Devices Corporation for free supplement with components.

References


About Authors...

Jiří PETRŽELA was born in 1978. He received a MSc. degree in 2003 at the University of Technology Brno. He is currently a PhD. student at the same university. His research interest is in design of nonlinear circuits and optimization of chaotic oscillators.

Václav POSPÍŠIL was born in 1978. He received a MSc. degree in 2003 at the University of Technology Brno. He is currently a PhD. student at the same university. His research interest is in modeling and solution of nonlinear circuits.