Nonlinear Resistor with Polynomial AV Characteristics and Its Application in Chaotic Oscillator

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Abstract. *This paper shows the realization of two terminal devices with an arbitrary polynomial nonlinearity up to the fifth order. The proposed design procedure is completely systematic using minimum of components. The very heart of our conception is four-channel four-quadrant analog multiplier MLT04.*

The implementation of synthesized nonlinear resistor as a general nonlinearity in chaotic oscillator is also presented and experimentally verified.

Keywords

Analog multiplier, chaos, polynomial nonlinearity, oscillator.

1. Introduction

The study of dynamical systems with nonlinear elements has been a topic of increasing interest for last two decades. This concern is especially obvious in the field of chaotic oscillators. However, there is no commercially available product with well-defined smooth nonlinearity. To eliminate this disadvantage several approaches were developed for synthesizing piecewise linear AV or transfer characteristics. These procedures are mostly based on controlled sources as VCVS, VCCS or negative impedance converters [2].

The behavior of a dynamical system is given by its equations and initial conditions. In some rare cases of the third-order dynamical systems, nor periodic, nor stable solutions occur when time goes to infinite. For these systems the chaotic attractor is observed. The possibility of chaos presence is about 0.5% and it grows with the dimension of studied system. The major property of chaotic system is in extreme sensitivity to tiny changes of initial state and long complex unpredictable behavior. We concentrate only on implicitly given differential equation of the type $\dddot{x} + a\ddot{x} + \dot{x} + x - f(x) = 0$, which exhibits chaotic behavior for a=0.6 and some piecewise linear, polynomial and even goniometrical functions $f(x)$. Integrating of each term of this equation uncovers that it is a damped harmonic oscillator driven by a nonlinear memory term involving the integral of $f(x)$. Systems of the form $\dddot{x} = F(\ddot{x}, \dot{x}, x)$ have been called jerk equations (time derivative of acceleration).

2. Design of a Nonlinear Resistor

It has been proved that analog multipliers can be used as a basic building block in resistors with monotonic AV characteristics. Using of the off-the-shelf device AD633 leads to a compact network structure with high sweep of input voltage ±11V and possible extension to higher-order polynomial. Transfer function of these IC is

$$W = K\left(X_1 - Y_1\right)\left(X_2 - Y_2\right) + Z \tag{1}$$

where K=0.1 denotes a scaling factor of an output voltage and X_i, Y_i, Z are independent input voltages.

The need of many IC since one package contains only one multiplier is the main disadvantage. For example, the fifth-order polynomial with all nonzero coefficients can be realized by five packages. This circuit is suggested in [4]. We can eliminate this drawback by using MLT04. This analog multiplier has maximal range of input voltage ±2.5V, low power supply ±5V and good temperature stability. Total linearity error is essential for our purpose and is typically only ±2% of the full scale. The primary static errors in an analog multiplier are input and output offset voltages. The input sources of errors can be eliminated by using the optional trim circuit. MLT04 includes four multipliers; each of them has its individual ground and realizes a simple transfer function

$$W_i = K\,X_i\,Y_i, \qquad i = 1, 2, 3, 4\,. \tag{2}$$

The driving point characteristics is given as the 5th order polynomial of the form

$$i = f(u) = a_5\,u^5 + a_4\,u^4 + a_3\,u^3 + a_2\,u^2 + a_1\,u + a_0\,. \tag{3}$$

The elementary operation in this manner is squaring. A compensation of internally set constant K=0.4±0.02 isn't allowed, although this factor is cumulated in higher-order coefficients of the polynomial.

Let's specify the positions of switches as S_i=±1. Then individual coefficients of (3) can be easily derived from the schematic in Fig. 9 as follows

$$a_0 = \frac{5}{R_5}\left(\frac{2R_{a2}}{R_a} - 1\right), \quad a_1 = \sum_{i=0}^{5}\frac{1}{R_i} - \frac{1}{R_6}, \quad a_2 = -S_1\frac{K}{R_1},$$

$$a_3 = -\prod_{i=1}^{2} S_i \frac{K^2}{R_2}, \; a_4 = -\prod_{i=1}^{3} S_i \frac{K^3}{R_3}, \; a_5 = -\prod_{i=1}^{4} S_i \frac{K^4}{R_4} . \quad (4)$$

where R_{a2} is a part of a variable resistor connected to –5V, which is supposed to be linear due to the average value of coefficient a_0 close to zero. A voltage follower is here employed as an isolation buffer. The input resistance of the invertor (consisting of the operational amplifier U1 and two affiliated resistors) has a negligible effect on final characteristics. Some examples of nonlinearities are demonstrated in Fig. 1. Our concretization is not convenient for approximation of goniometrical functions beyond one period. Let's recall the Taylor's power row formulae for sine function

$$\sin x = \sum_{n=1}^{\infty} \frac{1}{(2n-1)!}(-1)^{n+1} x^{2n-1} . \quad (5)$$

We can do this after using many multiplication cells.

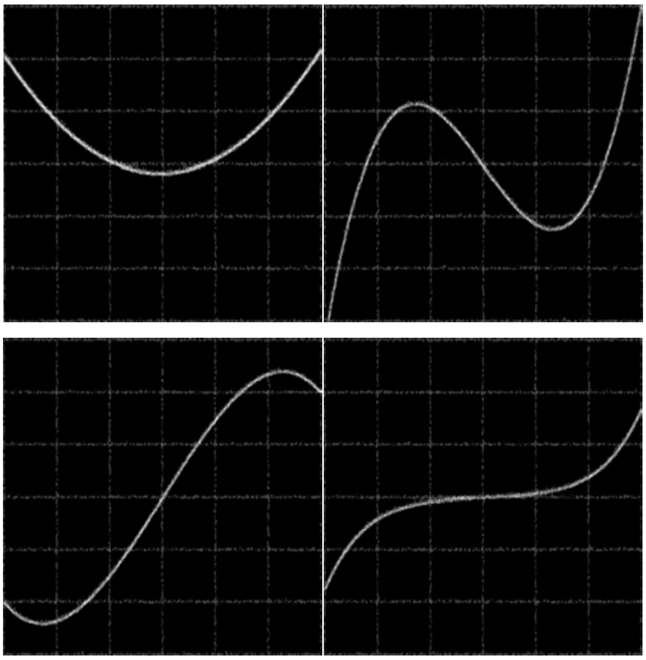

Fig. 1. Measured nonlinearities specified in **Tab. 1**.

The driving point characteristics can be measured by auxiliary triangular wave source and current sensitive 100Ω resistor. The digital oscilloscope grid is uniform for x-axis 800mV/div and flexible for y-axis 1, 3, 3, 5mA/div. The origins of these AV characteristics are screen centered. For complicated AV curves with many contraflexures, there is better to use DSP coupled with A/D and D/A converters. Moreover, such nonlinearity can be programmable. The operational amplifier U1 (in Fig. 10) and associated resistors form negative impedance converter with current inversion called INIC. This network has the effect of reflecting AV curve about the horizontal axis. It is equivalent to change the signs of all coefficients in (4).

3. Universal Chaotic Oscillator

In this section we discuss the circuitry implementation of mentioned chaotic oscillator. It is well known that 3rd order differential equation can be rewritten into the matrix form, i.e. into the set of 1st order differential equations. Our system being recasted has the form similar to dynamical systems of class C [6], in detail

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ q_3 & -q_2 & q_1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot f(x) \cdot \quad (6)$$

Due to the dissipativity of (6) for bounded chaotic attractor any continuous function $f(x)$ must have at least one zero at $x = \tilde{x}$. The stability of this fixed point is determined by the solution of the equilibrium problem $\det(\lambda \mathbf{E} - \mathbf{J}) = 0$, where **E** is unity matrix and **J** denotes Jacobi matrix

$$\lambda^3 - q_1 \lambda^2 + q_2 \lambda - q_3 - \frac{\partial f(x)}{\partial x} = 0 \quad (7)$$

evaluated at $x = \tilde{x}$. This point is then locally stable for (8a) and undergoes a Hopf bifurcation (complex conjugate eigenvalues cross the imaginary axis) if (8b)

$$q_1 q_2 < q_3 + \frac{\partial f(x)}{\partial x} < 0 \qquad q_1 q_2 - q_3 = \frac{\partial f(x)}{\partial x} . \quad (8a, b)$$

The transformation that preserves $\partial f(x)/\partial x$ and the graph of $f(x)$ only affects the size of the attractors. As follows from (6) our dynamical system deals with equivalent eigenvalue parameters q_1= -0.6, q_2= 1 and q_3= -1. In spite of its algebraical simplicity another modification can be applied. Huge numerical searches reveal chaotic cases with q_1= 0, namely the dual volume conserving cases with remaining parameters q_2= 4, q_3= -1. From the chaos point of view the principal quantity is Lyapunov exponent, which measures the divergence of two neighboring trajectories. The Poincaré-Bendixon's theorem implies the necessity of three state variables so there must be just three Lyapunov exponents. According to the natural definition of chaos one exponent must be positive and one equals to zero. The last one must be negative and its absolute value must be the largest as the attractor is loosing its energy. The variety of feedback functions, eigenvalues as well as base-e largest Lyapunov exponent (marked as LE) are documented in Tab. 1. Note that at least one equilibrium point is of the type of unstable spiral.

For a double scroll attractor only two sets of coefficients in (3) are important, namely $a_5 = a_4 = a_2 = a_0 = 0$ and $sign(a_3) \neq sign(a_1 + q_3)$ It is because of restriction on existence of three equilibrium points

$$\tilde{x}_1 = \begin{pmatrix}0 & 0 & 0\end{pmatrix}^T \qquad \tilde{x}_{2,3} = \begin{pmatrix}\pm\sqrt{\dfrac{-a_1-q_3}{a_3}} & 0 & 0\end{pmatrix}^T . \quad (9)$$

The dynamical systems producing a single folded band structure of attractors have quadratic nonlinearity $a_5 = a_4 = a_3 = 0$, $a_1 = -q_3$, $sign(a_2) \neq sign(a_0)$ and two equilibrium points

$$\tilde{x}_{1,2} = \begin{pmatrix}\pm\sqrt{-\dfrac{a_0}{a_2}} & 0 & 0\end{pmatrix}^T . \quad (10)$$

Individual trajectories were also numerically simulated by means of MathCAD 2000 using the 4th order Runge-Kutta iteration method with time interval (0, 300), iteration step 0.03 and initial conditions (0.1, 0, 0).

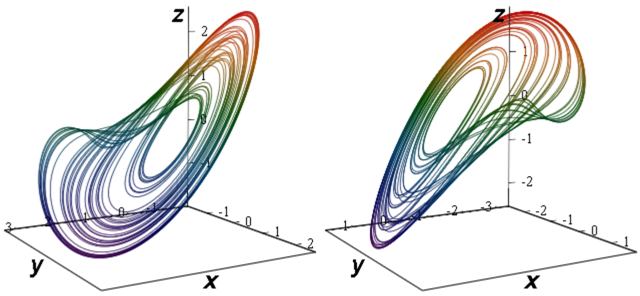

Fig. 2. State trajectories of both single folded band attractors.

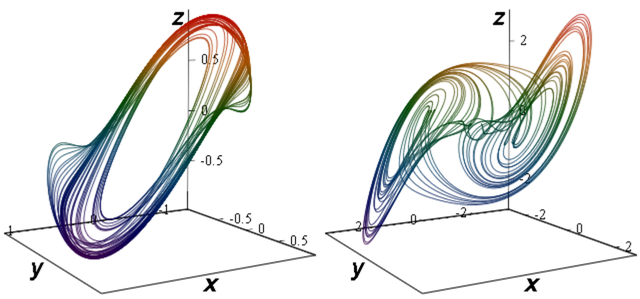

Fig. 3. Torus and double scroll chaotic attractors.

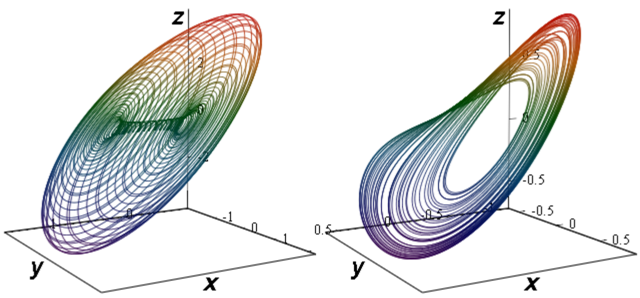

Fig. 4. Quasiperiodic orbits.

nonlinear $f(x)$	LE	eigenvalues at equilibria
$\pm 0.4x^2 \mp 0.2$	0.073	$-0.464, -0.068 \pm i0.965$
		$-1.056, 0.228 \pm i1.196$
$1.6x^3 - 0.6x$	0.103	$-1.067, 0.234 \pm i1.202$
		$1.107, -0.854 \pm i1.47$
$-0.177x^3 + 1.85x$	0.103	$0.531, -0.566 \pm i1.132$
		$-1.098, 0.249 \pm i1.219$
$x^n \quad n = 2, 3, 4, 5$	0.002	$-0.836, 0.118 \pm i1.088$
		$0.588, -0.594 \pm i1.16$
		$0.874, -0.737 \pm i1.321$
$x^2 + x - 0.29$	0.018	$0.616, -0.608 \pm i1.175$
		$-0.872, 0.136 \pm i1.104$

Tab. 1. Topology of attractors.

The linear part of (6) can be intuitively constructed by a cascade connection of elementary inverting integrators. Then terms in the last equation are realized by feedback and summing amplifier. However, this design procedure is definitely not the simplest. Another way is demonstrated in Fig. 10. We can employ the so called Antoniou's general impedance converter as high order admittance

$$Y(s) = L C_1 C_2 s^3 + R_g C_1 C_2 s^2 . \quad (11)$$

Further details are presented in [1]. State variable x is then chosen to be the voltage on the linear capacitor C_3 and state variable y is a current that flows through it. If both functional blocks are connected in parallel the proposed state equation is

$$f(u) = C_3 \frac{du}{dt} + R_g C_1 C_2 \frac{d^2u}{dt^2} + L C_1 C_2 \frac{d^3u}{dt^3} . \quad (12)$$

By this direct approach we obtain a circuit with unrealistic parameter values, namely resistors in Ohms, capacitors in Farads and inductance in Henries. Although it is possible to replace an inductor with its floating synthetic equivalent, such network has a drawback in two GIC requirements as it is illustrated in Fig. 11. Anyway, we are forced to introduce current r_c=1000 and time r_t=10000 rescaling. While doing it, resistors R_x alike as capacitors C_x and inductor L_x change their values

$$R_{res} = r_c R_x = 10^3 R_x \qquad L_{res} = r_c r_t^{-1} L_x = 0.1 L_x$$
$$C_{res} = r_c^{-1} r_t^{-1} C_x = 10^{-7} C_x \quad (13)$$

Some of circuit parameters can be fixed

$$R_f = 1\,k\Omega , \qquad C_1 = C_2 = C_3 = 100\,nF .$$

To preserve integration speed and chaotic behavior the rest of the GIC parameters should be computed

$$R_g = 900\,\Omega\,, \qquad L = 300\,mH\,.$$

On the contrary to $R_a = 100\ k\Omega$, for variable resistors $R_i = 1\ M\Omega$ a logarithmic RΔL characteristics is expected. Classical LM1458 can be employed to provide standard mathematical operations. It is cheap and commercially available device which contains two individual operational amplifiers. Sufficient performance in frequency domain guarantees trustworthy state trajectories with quick and repeated bursts.

PSpice 9.0 verification with *Final Time*=200ms and *Step Ceiling*=20μs of single folded band structure is shown in Fig. 5. This is the case with omitted INIC (S→1) and other values are

$$R_0 = 1\ M\Omega \quad R_1 = 1 k\Omega \quad R_2 = 1 M\Omega \quad R_3 = 1 M\Omega$$

$$R_4 = 1\ M\Omega \quad R_5 = 1 k\Omega \quad R_6 = 1 M\Omega \quad R_{a2} = 80\,k\Omega$$

Directly from (4), it is not hard to establish the value of R_6 according to completely eliminating of term $a_1 = 0$

$$R_6 = \frac{\prod_{i=0}^{5} R_i}{\sum_{i=0}^{5} \delta_i}\,, \qquad \delta_i = \prod_{\substack{j=0 \\ j \neq i}}^{5} R_j\,. \tag{14}$$

The state trajectory for the double scroll attractor is in Fig. 6. In this case, the switch S is in position 1 and the values of the resistors are

$$R_0 = 1\ M\Omega \quad R_1 = 1 M\Omega \quad R_2 = 900\,\Omega \quad R_3 = 1 M\Omega$$

$$R_4 = 1 M\Omega \quad R_5 = 1\ M\Omega \quad R_6 = 1.5\,k\Omega \quad R_{a2} = 50\,k\Omega\,.$$

Without INIC we obtain the following listing

$$R_0 = 1\ M\Omega \quad R_1 = 1 M\Omega \quad R_2 = 900\,\Omega \quad R_3 = 1 M\Omega$$
$$R_4 = 1 M\Omega \quad R_5 = 1\ M\Omega \quad R_6 = 600\,\Omega \quad R_{a2} = 50\,k\Omega$$

A torus is simulated in Fig. 7 with the switch in S→1 and the parameters of the resistors

$$R_0 = 1\ M\Omega \quad R_1 = 1 M\Omega \quad R_2 = 900\,\Omega \quad R_3 = 1 M\Omega$$

$$R_4 = 1\ M\Omega \quad R_5 = 1 M\Omega \quad R_6 = 4.1\,k\Omega \quad R_{a2} = 50\,k\Omega$$

Other quasiperiodic system can be displayed when the switch is in the position 1 and the values of the resistors

$$R_0 = 7\ k\Omega \quad R_1 = 400\,\Omega \quad R_2 = 1 M\Omega \quad R_3 = 1 M\Omega$$

$$R_4 = 1\ M\Omega \quad R_5 = 8\,k\Omega \quad R_6 = 500\,\Omega \quad R_{a2} = 40\,k\Omega$$

The good final agreement should be recognized by comparing individual trajectories simulated by using PSpice 9.0 with numerical results in Fig. 2, 3 and 4.

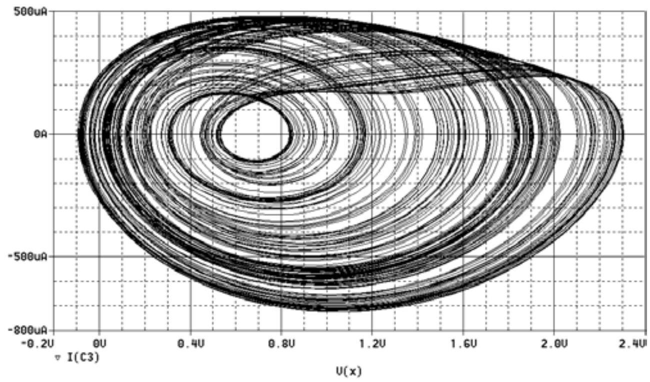

Fig. 5. I(C3)-V(C3) plane projection of the single scroll attractor.

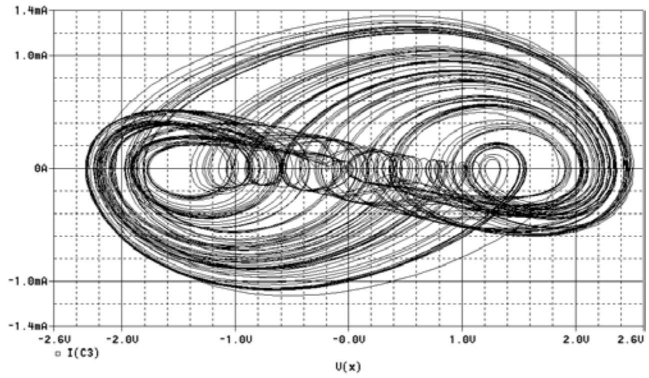

Fig. 6. I(C3)-V(C3) plane projection of the double scroll attractor.

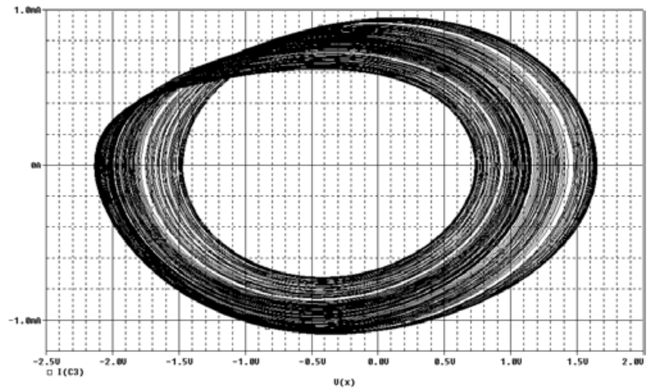

Fig. 7. I(C3)-V(C3) plane projection of the torus.

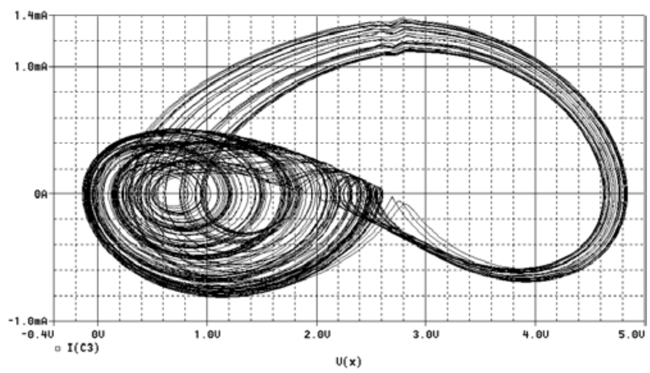

Fig. 8. I(C3)-V(C3) plane projection of the transition event.

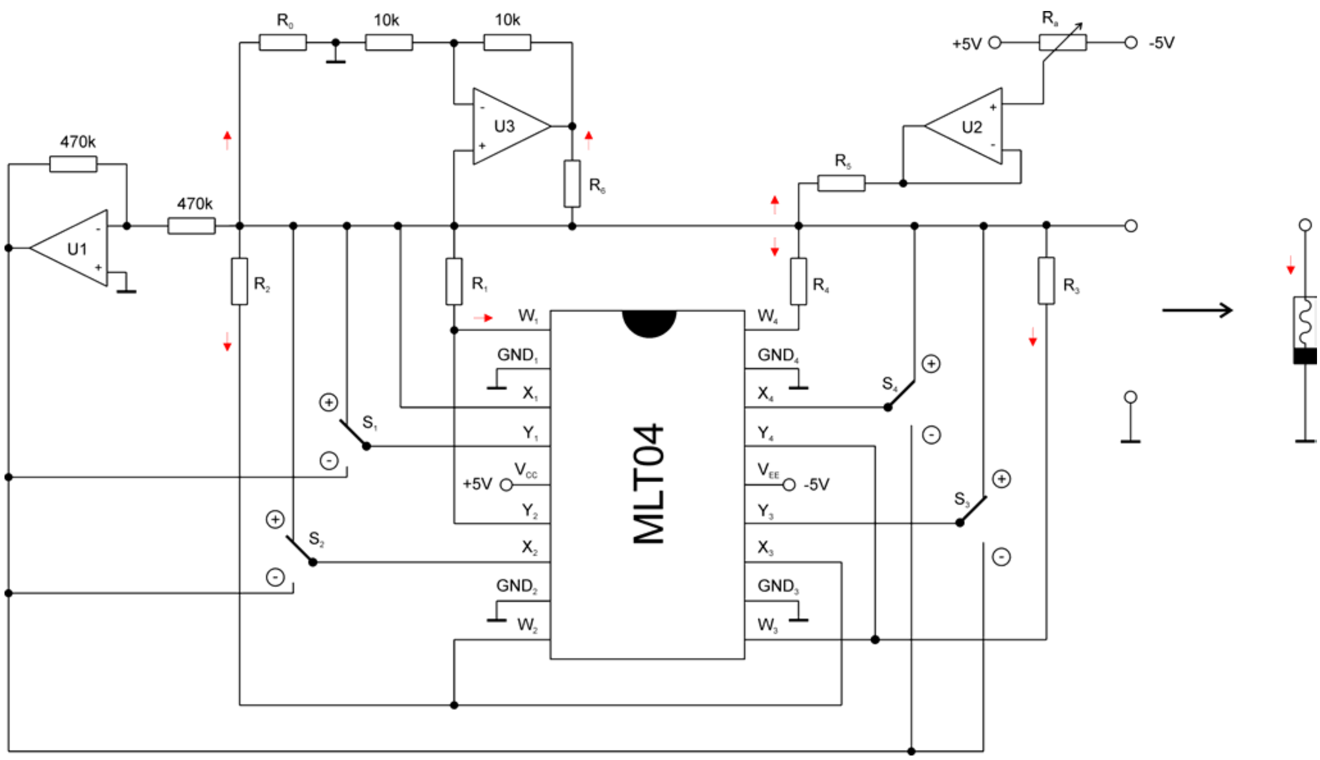

Fig. 9. Circuit implementation of polynomial resistive element (with current arrow convention).

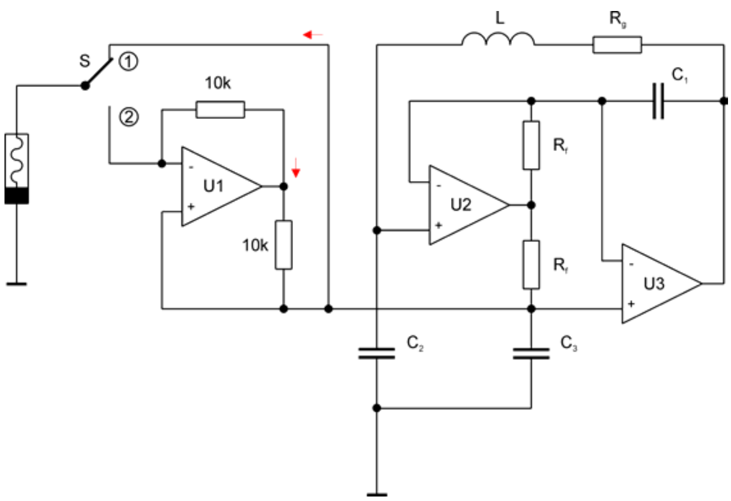

Fig. 10. Circuit implementation of universal chaos generator.

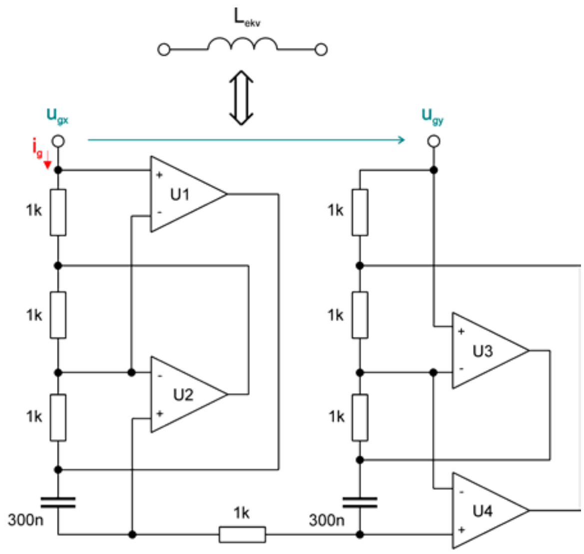

Fig. 11. First generation floating inductor replacement.

Fourier spectrum of chaotic signals is continuous and limited. The biggest distinguishable frequency component is about 5 kHz, so it is well into the audio range and many nonlinear phenomena can be heard. Bifurcation sequence goes through period doubling and can be easily traced. The evolution of the double scroll attractor, that means the process of merging two single scroll structures together, has been also watched (see Fig. 8). It stands to reason that for solving the dynamical system with $q_1 = 0$ the lossy inductor must be interchanged with an ideal one. Frequency spectrum is wider, spread roughly from zero to 100 kHz. Kirchhoff's laws give the equivalent inductance

$$u_{gx} - u_{gy} = i_g\, sCR^2 \quad \rightarrow \quad L_{ekv} = 300\, mH \tag{15}$$

4. Conclusion

Although the limitations are evident, the circuit implementation of universal chaotic oscillator with smooth nonlinearity was developed. This circuit works in hybrid voltage/current mode. Soft tuning of circuit parameters allows us to observe different structures of chaotic attractors. Physical realization would provide us with interesting tool for laboratory experiments. Realization could be made with using only four IC. INIC is not necessary for the correct function of our circuit, but it is well suited for quickly swapping between consumption and distribution regime of preadjusted nonlinearity.

Undoubtedly, chaotic oscillators will play the significant role in future communication techniques. Such system that produces chaotic trajectories with several shapes can be very useful, mostly in coders, modulators and broadband signal generators. In these applications it is supposed that nonlinear resistor simplifies into the form with two or three multiplication cells. For dynamical systems (6) we can deduce the mathematical model of synchronization based on modern control theory, as it is briefly discussed in [16].

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