Asymptotic Analysis of Optimal Piecewise Uniform Polar Quantization

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Abstract. *In this paper, a simple and complete asymptotical analysis is given for a mean square error (MSE) piecewise uniform polar quantizer (PUPQ). We show that PUPQ has the same performance as the asymptotic nonuniform polar quantizer (NPQ) and has implementation complexity between complexities of NPQ and uniform polar quantization. The goal of this paper is solving the quantization problem in case of PUPQ and finding the corresponding support region.*

Keywords

Polar quantization, asymptotical analysis, analytical and numerical optimization, distortion.

1. Introduction

Polar quantization techniques as well as their applications in areas such as computer holography, discrete Fourier transform encoding, image processing and communications have been studied extensively in the literature. Synthetic Aperture Radars (SARs) images can be represented in the polar format (i.e., magnitude and phase components) [1]. In the case of MSE quantization of a symmetric two-dimensional source, polar quantization gives the best result in the field of the implementation [1]. The motivation behind this work is to maintain high accuracy of phase information that is required for some applications such as interferometry and polarimetry, without loosing massive amounts of magnitude information [1]. Uniform polar quantizers are employed in Synthetic Aperture Radars (SARs) imaging systems, interferometric and polarimetric applications [1, 2]. Optimal uniform quantization is given in [3], but optimal quantizer is nonuniform. Generalize of uniform polar quantizer is a piecewise uniform polar quantizer. One of the most important results in polar quantization is due Swaszek and Ku who derived the asymptotically optimal nonuniform polar quantization [4]. The approximation given by Swaszek and Ku for the Nonuniform Polar Quantization (for reconstruction and decision levels $\mathbf{m} = (m_1, \ldots, m_L)$, $\mathbf{r} = (r_1, \ldots, r_{L+1})$ [4] is not correct because $r_{L+1} - m_L \to \infty$. That is the elementary reason for introducing support region r_{max}.

The support region for scalar quantizers has been found in [5] by minimization of the total distortion D, which is a combination of granular D_g and overload D_o distortions, $D = D_g + D_o$.

The nonlinear compressor characteristic is used in the paper [4]. Although the smooth and differentiable compressor characteristic is convenient for mathematical manipulations, there are problems of accurately implementing analog nonlinearities. Today's technology allows uniform quantizers or piecewise linear compressor characteristics implementation. In this paper, we give the simplest piecewise uniform quantization and show that it has approximately same performances as NPQ but it's much simpler for application.

The goal of this paper is solving the quantization problem in case of piecewise uniform polar quantizer and finding the corresponding support region. It is done by analytical optimization of the granular distortion and numerical optimization of the total distortion. A piecewise uniform polar quantizer consists of L different uniform polar quantizers. Different quantizers, however, may have different step-sizes. More precisely, our quantizer divides the input plane into L regions and every region is further sub-divided into L_i ($1 \le i \le L$) sub-regions. Each concentric ring in polar plane is allowed to have a different number of sub-partitions in the phase quantizer. The number of sub-partitions are denoted by N_{ij} , $1 \le i \le L$, $1 \le j \le L_i$. We perform the two-step optimization as in [6]: 1) the distortion optimization $D(\, i)$ in every partition under the constraint

$$\sum_{j=1}^{L_i} N_{i,j} = N_i \ ,$$

and 2) the optimization of the total granular distortion

$$\sum_{i=1}^{L} D(i) = D_g \ ,$$

which achieves the optimal distribution of sub-cells N_i on each sub-partition, under the constraint

$$\sum_{i=1}^{L} N_i = N \ .$$

We also gave the example of quantizer constructing for a Gaussian source. This case has the importance because of using Gaussian quantizer on an arbitrary source; we can take advantage of the central limit theorem and the known structure of an optimal scalar quantizer for a Gaussian random variable to encode a general process by first filtering it in order to produce an approximately Gaussian density, scalar-quantizing the result, and then inverse-filtering to recover the original [7].

2. Optimization

The probability density function (in polar coordinates) of a bivariate Gaussian random variable is [4]

$$f(r,\phi)=\frac{1}{2\pi\sigma^2}\,r\exp\left(\frac{-r^2}{2\sigma^2}\right)=\frac{f(r)}{2\pi}\ .$$

Without loss of generality, the variance can be assumed to be $\sigma^2 = 1$. Let us consider PUPQ with L partitions, each containing L_i sub-partitions. In order to minimize the total distortion we proceed as follows: We define the magnitude partition decision levels and reconstruction sub-partition levels as (see Fig. 1)

$$\begin{aligned}
&r_i=(i-1)\Delta; && 1\le i\le L;\quad r_{L+1}=r_{max}\\
&r_{i,j}=r_i+(j-1)\Delta/L_i; && 1\le i\le L,\quad 1\le j\le L_i+1;\, r_{L,L_{L+1}}=r_{L+1}=r_{max}\\
&m_{i,j}=r_i+(j-1/2)\Delta/L_i; && 1\le i\le L,\quad 1\le j\le L_i\,,
\end{aligned}$$

where $\Delta = r_{max} / L$. Let $\phi_{i,j,k}$ be the phase decision level, and let $\psi_{i,j,k}$ be k-th phase reconstruction level for the i-th partition and j-th sub-partition. Then $\phi_{i,j,k} = (k-1)\,2\pi / N_{i,j}$, $1 \le k \le N_{i,j} + 1$; and $\psi_{i,j,k} = (2k-1)\,\pi / N_{i,j}$, $1 \le k \le N_{i,j} + 1$.

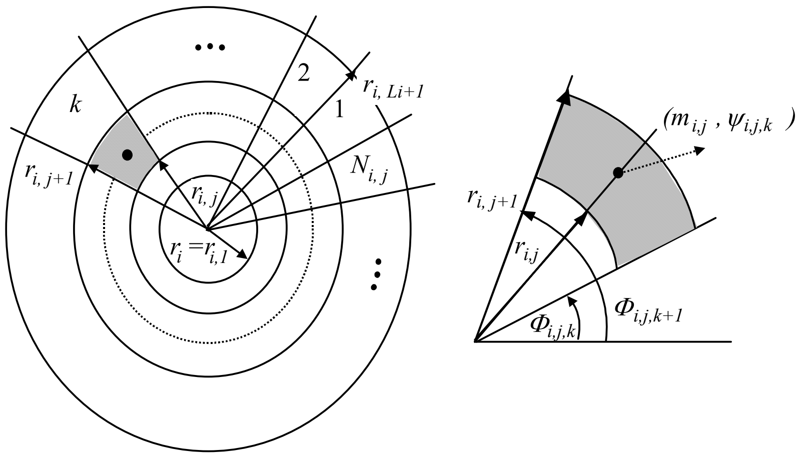

Fig. 1. The i-th partition of PUPQ and k-th cell on j-th level preview.

The total distortion per dimension is then

$$\begin{aligned}
D=\sum_{i=1}^{L}D(i)+D_o&=\frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{L_i}\sum_{k=1}^{N_{i,j}}\int_{\phi_{i,j,k}}^{\phi_{i,j,k+1}}\int_{r_{i,j}}^{r_{i,j+1}}\left[r^2+m_{i,j}^2-2rm_{i,j}\cos\left(\phi-\psi_{i,j,k}\right)\right]\frac{f(r)}{2\pi}\,dr\,d\phi\\
&+\frac{1}{2}\sum_{k=1}^{N_{L,L_L}}\int_{\phi_{L,L_L,k}}^{\phi_{L,L_L,k+1}}\int_{r_{max}}^{\infty}\left[r^2+m_{L,L_L}^2-2rm_{L,L_L}\cos\left(\phi-\psi_{L,L_L,k}\right)\right]\frac{f(r)}{2\pi}\,dr\,d\phi\ .
\end{aligned}\tag{1}$$

After the integration over ϕ and the reordering, the expression for D becomes

$$D=\sum_{i=1}^{L}\frac{1}{2}\left[\frac{\Delta_i^2}{12}P_i+\sum_{j=1}^{L_i}\frac{m_{i,j}^2\pi^2 f\left(m_{i,j}\right)}{3N_{i,j}^2}\Delta_i\right]+\frac{1}{2}\int_{r_{max}}^{\infty}\left(r-m_{L,L_L}\right)^2 f(r)\,dr+\frac{\pi^2}{6}\int_{r_{max}}^{\infty}\frac{r^2 f(r)}{N_{L,L_L}^2}\,dr\ .\tag{2}$$

To obtain (2), we use

$$\int_{r_{i,j}}^{r_{i,j+1}} rf(r)dr\approx m_{ij}f(m_{ij})\Delta_i;\ \Delta_i=\Delta/L_i\ ,$$

$$\sin(x)/x\approx 1-x^2/6\ .$$

In (2),

$$P_i=\int_{r_i}^{r_{i+1}} r\exp\left(-r^2/2\right)dr\ .$$

The optimization of $D(i)$ can be formulated in terms of Lagrange multiplier technique as follows. The optimization function is

$$J(i) = D(i) + \lambda \sum_{j=1}^{L_i} N_{ij} \ ,$$

where λ represents Lagrangian multiplier. Solving

$$\frac{\partial J(i)}{\partial N_{i,j}} = 0$$

under the constraint

$$\sum_{j=1}^{L_i} N_{i,j} = N_i$$

gives

$$N_{i,j} = \frac{N_i\, m_{i,j}\, \exp\left(-m_{i,j}^2/6\right)}{l_i} \Delta_i \ , \tag{3}$$

where

$$l_i = \sum_{j=1}^{L_i} \int_{r_{i,j}}^{r_{i,j+1}} r \exp\left(-r^2/6\right) dr \ .$$

Substituting (3) into $D(i)$, and from $\partial D(i) / \partial L_i = 0$ follows that

$$L_{iopt} = \sqrt[4]{\frac{P_i}{4 l_i^3}} \sqrt{\frac{N_i}{\pi}} \Delta \ . \tag{4}$$

Distortion $D(i)$ now becomes

$$D(i) = \frac{\pi}{6 N_i} (l_i^3 P_i)^{1/2} \ .$$

In order to minimize the function

$$D_g = \sum_{i=1}^{L} D(i)$$

under the constraint

$$\sum_{i=1}^{L} N_i = N = 2^{2R} \ ,$$

we use the method of Lagrange multipliers, where

$$J = D_g + \lambda \sum_{i=1}^{L} N_i \ .$$

From $\partial J / \partial N_i = 0$ we obtain

$$N_i = \frac{N \sqrt[4]{l_i^3 P_i}}{\sum_{k=1}^{L} \sqrt[4]{l_k^3 P_k}} \ , \tag{5}$$

and the expression for granular distortion becomes

$$D_g = \frac{\pi}{6N} \left(\sum_{i=1}^{L} \sqrt[4]{l_i^3 P_i} \right)^2 \tag{6}$$

Finally, replacing the summation operator in (6) by integration, we obtain distortion for NPQ [4]

$$D_g \approx \frac{\pi}{6N} \left[\int_0^{r_{max}} r \exp\left(-r^2/4\right) dr \right]^2 = \frac{2\pi}{3N} \left[1 - \exp\left(-r_{max}^2/4\right)\right]. \tag{7}$$

Hence, we have analytically derived an approximate equality of PUPQ and the best NPQ distortions. For $r_{max} = \infty$ PUPQ we get the same distortion as in [4]. Further distortion decrease can be achieved by the optimization of r_{max}. The exact optimal value for r_{max} is obtained by repeating our optimization method for different r_{max} and choosing the values for which $D = D_g + D_o$ is minimal.

As an illustration of PUPQ performance, in Fig. 2 we show signal-to-quantization noise ratio $SNR = 10 \log(1/D)$ as a function of the number of bits per sample R.

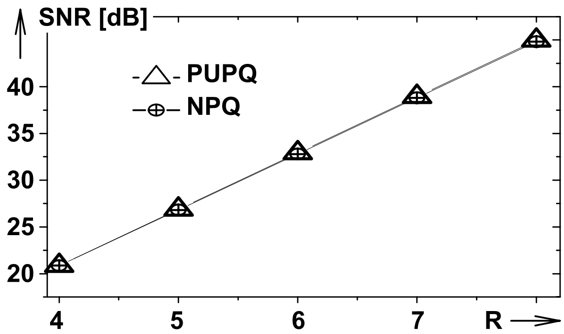

Fig. 2. PUPQ performances (SNR) versus rate for $L = 8$ compared to theoretical NPQ bound.

Obviously for a Gaussian source, the performances of the optimal PUPQ are the same as of the best NPQ [4]. However, comparing PUPQ results for higher L with those given in paper [4] we have that our results would be better than the best known. The reason of obtaining better results lies in a fact that the method we proposed defines optimal support region.

3. Conclusions

In this paper, we give the simplest piecewise uniform quantization and show that it has approximately the same performances as NPQ but it's much simpler for application. We give the equation for optimal number of points for different levels and, also, optimal number of levels for every partition. The equation for the D_g^{opt} is given in a closed form.

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Dr. František Židek – 70 Years Old

Dr. František Židek was born on the 10^{th} of July 1934 in Trnava, Slovakia. In 1957, he graduated with honors at the Slovak University of Technology in Bratislava. He obtained his degree of Eng. (M.S.) in the field of Telecommunication. In 1969, he received the degree of Doctor (Ph.D.) majoring in Radio Electronics under the supervision of Prof. Ján Chmúrny. In 1977, he reached the title habilitant docent.

In 1965, Dr. František Židek spent six months at the Cornell University, Ithaca, N.Y., USA as a research student, where he studied the Information Theory under supervision of Prof. F. Jelinek. He spent the years 1969 to 1971 at the Tokyo Institute of Technology in Japan. He worked in Prof. Yana-Gisawa's laboratory. His interest at that time was focused in the synthesis of linear active circuits. During the years 1985 to 1988, Dr. František Židek was invited to give lectures at the Higher Institute of Electronics, Beni Walid, Libya. He was also invited to give some lectures for the university staff of universities in many other countries. For a short time period, he was also a guest of many Universities all over the world.

For many years, Dr. František Židek supervised the Signal Processing Section of the Department of Radio Electronics, Slovak University of Technology (SUT) in Bratislava, and later, he was the head of the same department. He was the member of the Scientific Council of the Faculty of Electrical Engineering and Information Technology, SUT in Bratislava. He was also the member of the Editorial Board of two journals: Radioengineering, and Journal of Electrical Engineering.

Educational activities of Dr. František Židek are very extensive. He introduced 14 new modern courses into the curriculum. He wrote 17 textbooks for students, three of them were written in English. Dr. Židek was the member of the State Committee for the Doctoral Degree Final Examinations in Slovakia and he was also the member of the State Committees for the Engineering Degree Final Examinations at the Universities in Bratislava and Košice.

Research activities of Dr. František Židek are focused in three main areas: reliability and renewal theory, active circuit theory, and signal processing. He published many original papers, large research reports and technical reports of applied research for the industrial use. He cooperated with many domestic and foreign universities, with the Academy of Sciences in Prague, with research and industrial institutes in the Czech Republic and the Slovak Republic.

His scientific contributions are devoted mostly to the occurrence of non-stationary random early failures of electronic systems. In last years, he published papers on time-frequency representation of non-stationary random signals or processes. In all the above-described specializations, Dr. Židek supervised over 60 graduate students.

Editorial Board of Radioengineering Journal and all staff of Departments of Radio Electronics in the Czech Republic and in the Slovak Republic congratulate him to his jubilee. Best wishes, healthy and prosperous further years.

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