A New 3-D Piecewise-Linear System for Chaos Generation

Zeraoulia ELHADJ

Dept. of Mathematics, University of Tébessa, 12000, Algeria

zeraoulia @ mail.univ-tebessa.dz, zelhadj12 @ yahoo.fr

Abstract. *We propose in this paper a new simple continuous-time piecewise-linear three dimensional system for chaos generation. Nonlinearity in this model is introduced by the characteristic function of the Chua's circuit given in* [1]. *Simulated results of some chaotic attractors are shown and justified numerically via computing the largest Lyapunov exponent. The possibility and the robustness of the circuitry realization is also given and discussed.*

Keywords

Piecewise-linear system, new chaotic attractor, circuitry realization, Chua's diode.

1. Introduction

Piecewise linear chaotic systems are used to construct simple electronic circuits with several applications in electronic engineering, in particular secure communications [6 to 10]. This motivates the present study of the problem of generating new 3-D chaotic attractor using the characteristic function of the Chua's circuit [1] as the nonlinearity of the proposed system. Some basic dynamical behaviors of the new system are also investigated in the present paper.

Electronic circuits provide very useful systems for studying chaos as well as easy applications. Chua's circuit provides an ideal paradigm for research on chaos and bifurcation by means of theoretical analysis and laboratory experiments. This circuit, presented by L. O. Chua [1], contains four linear elements and a nonlinear one (Chua's diode). This nonlinear element is characterized by a piecewise-linear function, characteristic of Chua's resistor whose functional representation in dimensionless form is expressed by:

$$h(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| - |x-1|)$$

The Chua's system with this nonlinearity gives a chaotic attractor called the Double scroll attractor [5]. In this work we use only the piecewise-linear function as the nonlinear element in a very simple constructed dynamical system; recently we discovered that the proposed system generates a new chaotic attractor. This statement is very important because it is possible to build simple electronic circuits, e.g., with piecewise-linear function, to realize chaos in differential systems [2], [3], [4], [6], because there are several possible practical implementations of the Chua's diode, by using off-the-shelf components: diodes, transistors, op-amps [7]. An IC chip implementation of the whole Chua's circuit was realized by using a 2μm CMOS technology [8].

2. The Proposed Chaotic Oscillator

Our proposed model for chaos generation is given by the following third-order continuous-time piecewise-linear system:

$$\begin{aligned} dx/dt &= -ax + y + cz, \\ dy/dt &= -x - ay, \\ dz/dt &= h(y) - z, \end{aligned} \tag{1}$$

where h is the characteristic function of the Chua's equation given above and a, c are the bifurcation parameters. Despite the extreme simplicity of system (1), it is capable of producing a variety of dynamical behaviors depending on the choice of the bifurcation parameters.

3. Some Basic Properties

Some basic dynamical behaviors of the chaotic system (1) are investigated here by both theoretical analysis and numerical simulation.

3.1 Symmetry and Dissipativity

System (1) has a natural symmetry under the coordinates transform $(x,y,z)\to(-x,-y,-z)$ which persists for all values of the system parameters. In the following assume that $m_0<0$, $m_1<0$ and $c>0$. Therefore for system (1), the divergence of the flow is given by:

$$\nabla V = \frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} = -(2a+1).$$

Then, if $a > -½$ the system (1) has a bounded globally attracting ω-limit set. Hence, this system is dissipative. Thus, all trajectories ultimately are confined to a specific subset having zero volume and the asymptotic motion settles onto an attractor, this result has been confirmed by some computer simulations. Notably, the system (1) is not equivalent to the mentioned Chua system, because it is straightforward to verify that there is no non-singular (linear or nonlinear) coordinate transforms (i.e. diffieomorphism) that can convert system (1) to Chua's system given in [1] or vice versa.

3.2 Equilibrium Points and Their Stability

In this section assume that $m_0 < 0$ and $a^2+1+cm_1 \neq 0$

Due to the shape of the vector field of system (1) the phase space can be divided into three piecewise-linear regions denoted by D_i, $i = -1, 0, 1$ as follow:

$$\begin{cases} D_1 = \{(x,y,z) / y \geq 1\}, \\ D_0 = \{(x,y,z) / |y| \leq 1\}, \\ D_{-1} = \{(x,y,z) / y \leq -1\}. \end{cases}$$

In each of these regions, there exists a "symmetric" point $P^-(-x,-y,-z)$ for each equilibrium $P^+(x,y,z)$, due to the symmetry of the vector field.

The equilibria of the system (1) are found by solving the three equations: $x'=y'=z'=0$ which gives:

(1) If $c \geq \frac{-(a^2+1)}{m_0}$ then the system (1) has 3 equilibria:

$P^{\pm} = (\pm\delta, \mp a\delta, \pm h(\delta))$, and $P^0 = (0,0,0)$; where

$$\delta = \frac{-c(m_0 - m_1)}{a^2 + 1 + c\, m_1}, \quad P^+ \in D_1, P^- \in D_{-1}, P^0 \in D_0.$$

(2) If $c < \frac{-(a^2+1)}{m_0}$, then the system (1) has 1 equilibrium point $P^0 = (0,0,0)$.

Let us now study their stability. For this the eigenvalues are the solution of characteristic cubic equation: $P(\lambda)=\lambda^3+A\lambda^2+B\lambda+C$. The values of A, B, and C are determined for each equilibrium point as follows, for $P^{\pm}$ one has: $A_1=2a+1$, $B_1=2a+a^2+1$ and $C_1=a^2+1+cm_1$, and for P^0 one has: $A_0=A_1$, $B_0=B_1$ and $C_0=a^2+1+cm_0$.

The Routh-Hurwitz conditions lead to the conclusion that the real parts of the roots λ are negative if and only if: $A>0$, $C>0$, and $AB-C>0$. It is remarked that the equilibrium points $P^{\pm}$ of system (1) have the same type of stability, and since $c \geq -(a^2+1)/m_0$ one has $C_0C_1<0$; then P^0 and $P^{\pm}$ have different topological type. P^0 is always unstable and therefore has at least one real and positive eigenvalue. On the other hand, the exact value of the eigenvalues is obtained by using the Cardan methods for solving a cubic equation. Also, with $a>0$, $c=g(a)/m_1$ is a Hopf bifurcation point for $P^{\pm}$ and with $a>0$, $c=g(a)/m_0$ is a Hopf bifurcation point for P^0, where $g(a)=2a(2+2a+a^2)$.

4. Dynamics Behaviors with Parameter Variation

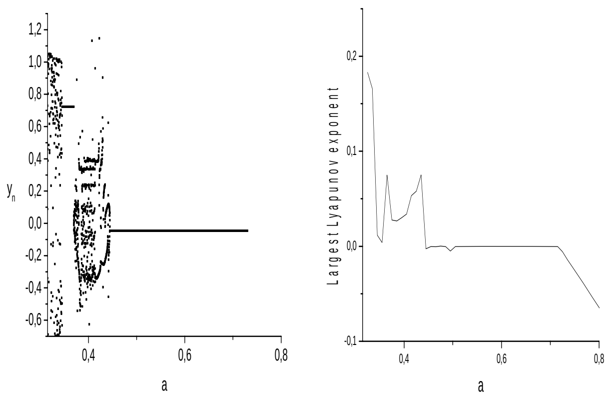
Now, the dynamical behaviors of the system (1) are investigated numerically where we use in each case an appropriate Poincaré section where the resulting points y_n are computed by using Hénon method, and a set of one of theme is recorded after transients have decayed and plotted versus the desired parameter. The calculations of limit sets of the system (1) were performed using a fourth Runge-Kutta algorithm with a constant step size $\Delta t=10^{-3}$, then, to determine the long-time behavior and chaotic regions, we numerically computed the largest Lyapunov exponent.

4.1 Variation of Parameter

Fix parameters $c = 4$, and let $a \geq -0.5$ vary. Then the system (1) exhibits the following dynamical behaviors:

- When $-0.5 < a < 0.315$ the system does not converge.
- When $0.315 \leq a < 0.345$ the system converges to a chaotic attractor (see Fig. 3).
- When $0.345 \leq a < 0.361$ there is a limit cycle, as shown in Fig. 5.
- When $0.361 \leq a < 0.379$ the system converges to a chaotic attractor (See Fig. 4).
- When $0.379 \leq a < 0.390$ there is a periodic window (See Fig. 5 and Fig. 6).
- When $0.390 \leq a < 0.420$ the system converges to a chaotic attractor (See Fig. 3).
- When $0.420 \leq a < 0.730$ there is a periodic window (See Fig. 5 and Fig. 6).
- When $a \geq 0.730$the system converges to a point.

For drawing the bifurcation diagram we used an appropriate Poincaré section Σ defined by: $\Sigma=\{(x,y)\in R^2/z = 0\}$. The result is shown in Fig.1 (a).



(a) (b)

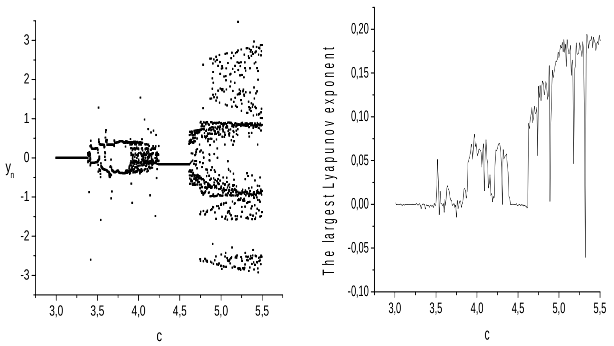
Fig. 1. **(a)** Bifurcation diagram of the variable y_n plotted versus the control parameter $0.315 \leq a \leq 0.8$, with $c = 4$. **(b)** Variation of the largest Lyapunov exponent of system (1) versus the parameter $0.315 \leq a \leq 0.8$, with $c = 4$.

4.2 Variation of Parameter *c*

Fix parameters $a = 0.4$ and let $c > 0$ vary. Then the system (1) exhibits the following dynamical behaviors:

- When $0 < c < 1.6$, the system converges to a point.
- When $1.6 \leq c < 4$, there is a periodic window (See Fig. 5 and Fig. 6).
- When $4 \leq c < 4.15$, the system converges to a chaotic attractor (See Fig. 3).
- When $4.15 \leq c < 4.1569$, the system converges to a cycle limit (See Fig. 6).
- When $4.1569 \leq c < 4.289$, the system converges to a chaotic attractor (See Fig. 4).
- When $4.289 \leq c < 4.61$, there is a periodic window (See Fig. 5 and Fig. 6).
- When $4.61 \leq c < 5.54$, the system converges to a chaotic attractor (See Fig. 3).
- When $c \geq 5.54$, the system does not converges.

For drawing the bifurcation diagram we used the same Poincaré section Σ defined above. The result is shown in Fig.2 (a).



(a) (b)

Fig. 2. **(a)** Bifurcation diagram of the variable y_n plotted versus the control parameter $3 \leq c \leq 5.54$, with $a = 0.4$. **(b)** Variation of the largest Lyapunov exponent of system (1) versus the parameter $3 \leq c \leq 5.54$, with $a = 0.4$.

One remarks that the dynamics of the system (1) with respect to c is a reverse scenario to the one given with respect to the parameter a.

In the following we present various numerical results to show the chaoticity of system (1), with the gives of the largest Lyapunov exponent (LE) as the usual test for chaos as shown in Figs. 1(b), 2(b). Thus, for: m_0= -0.43, m_1=0.41, $a = 0.4$, $c = 4.66$, the system (1) has the new chaotic attractor presented in Fig.1. All phase portraits are done in the following manner: (a) Projection into the *x-y* plane, (b) Projection into the *x-z* plane, (c) Projection into the *y-z* plane.

5. Possibility of the Circuitry Realization of the New System

In recent years, much work has been devoted to building simple electronic circuits, i.e. with piecewise linear function, to realize chaos in differential systems [3], [4], [6-10]. Since the nonlinearity of system (1) is a piecewise linear function and robust [7] and the remaining system elements are linear, then they should be confined to their linear regions of operation; this simplifies the circuitry realization of the new system (1).

6. Conclusions

This paper has reported the finding of a new simple piecewise linear three dimensional chaotic system. Basic properties of the system have been analyzed by means of the largest Lyapunov exponent and bifurcation diagrams of an associated Poincaré map. The synthesis of the electronic circuit which represents the proposed system has much in common with Chua´s oscillator.

References

[1] CHUA, L. O., KOMURO, M., MATSUMOTO, T. The double scroll family. Parts I and II, *IEEE Trans. Circuits. Syst*., 1986, 33, p. 1073-1118.

[2] SPROTT, J. C. Some simple chaotic flows. *Phys. Rev.* E50, R647650, 1994.

[3] CHUA, L. C., LIN, G. N. Canonical Realization of Chua's circuit Family. *IEEE Transactions in Circuit and Systems,*. 1990, vol. 37, no. 7, p. 885-902.

[4] KARADZINOV, L. V., JEFFERIES, D. J., ARSOV, G. L., DEANE, J. H. B. Simple piecewise-linear diode model for transient behaviour. *International Journal of Electronics*, 1996, vol.78, no. 1, pp 143 to 160.

[5] KOMURO, M., TOKUNAGA, R., MATSUMOTO, T., HOTTA, A. Global bifurcation analysis of the Double Scroll circuit. *Int. J. Bifurcation and Chaos*, 1991, vol. 1, no. 1, p. 139-182.

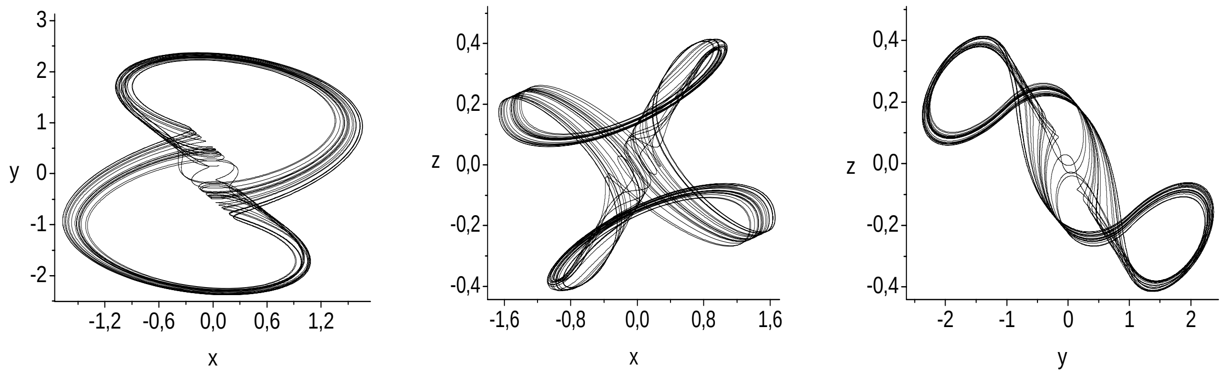
[6] BAHIOUS, E. H., JARRAY, P. Lorenz attractor from differential equation with piecewise linear terms. *Int.J. Bifurcation and Chaos, 1993,* vol. 3, no. 1, p. 201-210.

[7] KENNEDY, M. P. Robust OP Amp realization of Chua's circuit. *Frequenz,* 1992, vol. 46, no. 3-4, p. 66-80.

[8] CRUZ, J. M., CHUA, L. O. [1993] An IC chip of Chua's circuit. *IEEE Trans. Circuits Syst.-II: Analog and Digital Signal Processing,* 1993, 40(10), p. 614-625.

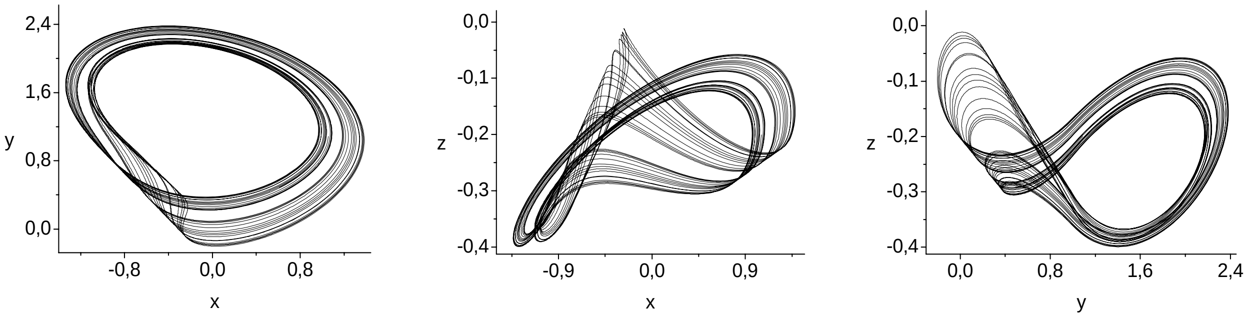
[9] SPROTT, J. C. A new class of chaotic circuit. *Physics Letters A*, 2000, 266, p. 19–23.

[10] SPROTT, J. C. Simple chaotic systems and circuits. *Am. J. Phys*., 2000, vol. 68, no. 8, p. 758-763.



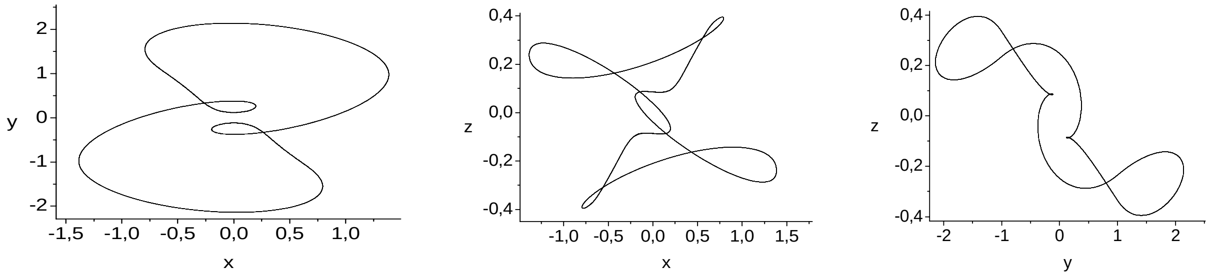
(a) (b) (c)

Fig.3. Phase portrait of system (1) with $a = 0.4$, $c = 4.66$ and in this case $LE = 0.381$.



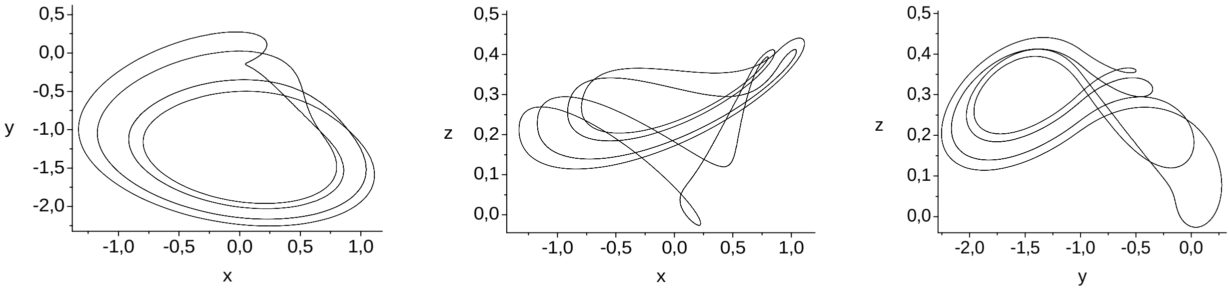
(a) (b) (c)

Fig.4. Phase portrait of system (1) with $a = 0.5$, $c = 5$ and in this case $LE = 0.379$.



(a) (b) (c)

Fig.5. Phase portrait of system (1) with $a = 0.345$, $c = 4$ and in this case $LE = 0$.



(a) (b) (c)

Fig.6. Phase portrait of system (1) with $a = 0.44$, $c = 4$ and in this case $LE = 0$.