On the Relationship between Integer Lifting and Rounding Transform

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Abstract. In this paper we analyze the relationship between integer Lifting scheme and Rounding transform as a means to compute the wavelet transform in signal processing area. We bring some new results which better describe relationship, reversibility and equivalence of integer lifting scheme and rounding transform concept.

Keywords
Wavelet, lifting, transform, integer, rounding.

1. Introduction

At present time, there is still an effort to introduce a new general concept of an integer-to-integer transform. The most known is the integer version of the lifting scheme [1], [2], [3] (see Fig. 1) - the integer lifting scheme (ILS) [1], [2], [3] (see Fig. 2). Lifting factorizes wavelet transform into successive invertible steps.

Besides the ILS, there is also a variety of transforms that operate as integer-to-integer transforms [10 - 13]. Although some of them are ad-hoc solutions, some have more generic construction. We’ve more closely examined the idea of Prost and Jung. Their concept is based on the floor operator applied after the forward transform and the ceil operator applied after the backward transform. They refer to this concept as the Rounding transform (RT) [4], [5], [7], [8]. The forward transform of RT is defined as following:

\[ y = \left\lfloor Hx \right\rfloor. \]  

(1)

The backward transform is defined:

\[ x = \left\lfloor Vy \right\rfloor \Leftrightarrow \left\lceil V\left\lfloor Hx \right\rfloor \right\rceil \]  

(2)

where matrix \( V \) is defined as inverse matrix to \( H \). Later they introduced the overlapped Rounding transform (ORT) [6], where terms in (1) and (2) contained polynomials. Despite of the interesting idea, the authors were unable to show general solution that would satisfy (2). Instead they presented several types of matrices that satisfy the mentioned equation. Later, these particular matrices have been revisited by Adams and Kossentini [14] and have been found lifting equivalent.

We have examined the relationship between integer lifting and its rounding counterpart (Fig. 3). First we show the set of equivalence, i.e. set where RT and ILS produce identical outputs for identical inputs with respect to \( p, u, p', u' \). This is shown in section 2. Then in section 3 we show, when the rounding transform is reversible and how important the relationship between the two approaches is.

2. Relationship between Integer Lifting and Rounding Transform

To show the relationship between ILS and RT, we’ve chosen a version of the ILS with one predict and update step (without the normalization step). This scheme is compared against its RT counterpart (rounding operation applied at the end of the forward transform) as it is shown in Fig. 3.
Although RT shown here is not a general RT (the H matrix for it is not a general one as we’ll show later), it is a sufficient example to demonstrate how RT and ILS are related. Because both transforms are integer-to-integer transforms and the result should be representable in a machine with finite precision, we will assume

\[ x_e, x_o \in Z \text{ and } p, u, p', u' \in Q. \]  

In appendix A it is shown, that there is no case, where both approaches would produce identical outputs for all possible (and identical) inputs, when \( p \neq p' \) and \( u \neq u' \). Then according to Fig. 3, the outputs for the ILS are

\[
\begin{bmatrix}
    y_1 \\
    y_2 
\end{bmatrix} = \begin{bmatrix} x_e + [u \{ px_e + x_o \}] \\
    [px_e + x_o]
\end{bmatrix}
\]  

and the outputs produced by the RT scheme are

\[
\begin{bmatrix}
    y_1' \\
    y_2'
\end{bmatrix} = \begin{bmatrix} x_e + [u \{ px_e + x_o \}] \\
    [px_e + x_o]
\end{bmatrix}
\]  

From Fig. 3., it is clear, that \( y_2 = y_2' \). To analyze the equivalence of ILS and RT, we’ll need to find all \( p, u \) satisfying the equation

\[ y_1 = y_1' \]  

for all possible \( x_e, x_o \). The equation to solve is:

\[ x_e + [u \{ px_e + x_o \}] = x_e + [u \{ px_e + x_o \}] \]  

According to (3), we can express \( u \) and \( p \) in the following form

\[ u = \frac{U_1}{U_2}, \quad p = \frac{P_1}{P_2} \quad \text{where} \]

\[ U_1, P_1 \in \mathbb{Z} \land U_2, P_2 \in \mathbb{N} - \{0\}. \]  

Equation (7) is satisfied only when

\[ P_1 = 1 \text{ for } U_1 < 0 \text{ or } \]

\[ 0 \leq \frac{U_2 - 1}{U_1} \left( \frac{P_1}{P_2} - 1 \right) < 1 \text{ for } U_1 \geq 0. \]  

We call the set of \( p \) and \( u \), where (9) holds, the set of equivalence. It is depicted in Fig. 4.

\[ P \]

Fig. 4. Set of equivalence of ILS and RT. White area depicts where the equation (9) holds and black area where it doesn’t. Displayed is range \( p, u \in (-5, 5) \).

### 3. Reversibility of Rounding Transform

To examine the reversibility of RT, we’ll have to examine the whole scheme, not only the forward part of it (as in section 2). The scheme in the Fig. 3b corresponds to (2) with the following forward and backward matrix:

\[ H = \begin{pmatrix} 1 + up & u \\ p & 1 \end{pmatrix}, \quad V = H^{-1} = \begin{pmatrix} 1 & -u \\ -p & 1 + up \end{pmatrix}. \]

Then for outputs \( \hat{x}_e \) and \( \hat{x}_o \) it holds:

\[ \begin{bmatrix} \hat{x}_e \\
    \hat{x}_o \end{bmatrix} = \begin{bmatrix}
    x_e + [u \{ px_e + x_o \}] - u \{ px_e + x_o \} \\
    x_e + [u \{ px_e + x_o \}] + u \{ px_e + x_o \} - [u \{ px_e + x_o \}]
\end{bmatrix}. \]

The reversibility is achieved when \( \hat{x}_e = x_e \) and \( \hat{x}_o = x_o \). Solution of (11) is not trivial, but a closer look on the reconstruction of \( x_e \) in ILS and RT approach reveals that it is done in the same way. Based on Fig. 3. for lifting

\[ x_e = y_1 + [-y_2 u] = y_1 - [y_2 u] \]

and rounding

\[ x_e = y_1' - [y_2 u] \]
Equation (12) always holds, since the integer lifting is reversible. Following this, (13) can hold only if \( y_1 = y' \).
Considering \( y_2 = y' \), this finally tells us: if the RT is reversible, then it is possible to write it in an ILS form.

4. Conclusion

From what we have shown in section 3, one can see that the forward part of RT is in fact ILS in our case. Nothing is said about the reconstruction of \( \hat{x}_y \). Solving \( \hat{x}_y = x_y \) from (11) is difficult, since the backward part of RT is different than in ILS. Following results that we’ve derived in section 3, it’s obvious that the set of reversibility will be not greater than the set of equivalence. Expansion of this statement to a polynomial case should be straightforward.

Appendix A - Equivalence of ILS and RT Approach

The output of ILS, according to Fig. 3, is
\[
\begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix} = \left( x_y + [u'[p_1 x_y + x_n]] \right)
\]
and for RT we have
\[
\begin{pmatrix}
  y_1' \\
  y_2'
\end{pmatrix} = \left( x_y + [u'[p_1 x_y + x_n]] \right).
\]
The equivalence is achieved when
\[
y_2 = y' \text{, i.e. } \left[ p_1 x_y \right] + x_n = \left[ p_1 x_y \right] + x_n, \tag{A1}
y_1 = y', \text{ i.e. } x_y + [u'[p_1 x_y + x_n]] = x_y + [u'[p_1 x_y + x_n]], \tag{A2}
\]
where \( p, u, p', u' \in Q \) (since working with finite precision). In the following we’ll use these properties:

1. \( \alpha = \frac{A_1}{A_2} \) where \( A_1 \in Z \land A_2 \in N \setminus \{0\} \),
2. for every \( Ax \) where \( A \in Q \), \( x \in Z \) we can write
\[
Ax = \frac{A_1}{A_2}x = C(x) + \frac{D(x)}{A_2}
\]
where \( D(x) \in \{0, \ldots, A_2 - 1\} \) and \( A_1 \in Z \land A_2 \in N \setminus \{0\} \).

Solution (A1)

We can rewrite \( p' \) as \( p' = p + \Delta_p \) so we obtain
\[
[p_1 x_y] = [p_1 x_y + \Delta_p x_n].
\]
It’s clear that the right term of (A1) will grow over all limits with nonzero \( \Delta_p \). The equation (A1) would not hold for all \( x \) in this case. Therefore, to fulfill (A1) we have to choose: \( \Delta_p = 0 \), which means \( p' = p \).

Solution (A2)

Instead of using \( p_1 x_y + x_n \), we’ll express it as
\[
x = p_1 x_y + x_n = (P_1/P_2)p_1 x_y + x_n \] This shows that \( x \) is either integer or an integer plus a fraction of \( P_2 \). This way, \( x \) can be expressed also as \( x = \lfloor x \rfloor + \Delta_x \) whereby \( \Delta_x \in \{0, 1, \ldots, (P_2 - 1)/P_2\} \). If we write \( u' = u + \Delta_u \) we can express (A2) as follows
\[
\begin{align*}
\left[ u \cdot x \right] & = \left[ u \cdot \lfloor x \rfloor + u \Delta_x + \Delta_x x \right], \tag{A3} \\
u \cdot x & \text{ can be expressed as}

\begin{align*}
\frac{u \cdot x}{P_2} & = \frac{\alpha(x) + \beta(x)}{U_2} + u \Delta_x + \Delta_x x. \\
\end{align*}
\end{align*}
\]
whereby \( \beta(x) \in \{0, 1, \ldots, U_2 - 1\} \). The (A3) changes then to
\[
\begin{align*}
\frac{\alpha(x)}{U_2} + \frac{\beta(x)}{U_2} & = \alpha(x) + \frac{\beta(x)}{U_2} + u \Delta_x + \Delta_x x, \tag{A4}
\end{align*}
\]
and since \( \frac{\beta(x)}{U_2} = 0 \) further
\[
\begin{align*}
0 & = \frac{\beta(x)}{U_2} + u \Delta_x + \Delta_x x. \tag{A5}
\end{align*}
\]
Substituting one more time \( x = \lfloor x \rfloor + \Delta_x \) we obtain
\[
\begin{align*}
\left[ \frac{\beta(x)}{U_2} + \Delta_x (u + \Delta_u) + \Delta_u x \right] & = 0. \tag{A6}
\end{align*}
\]
That means, that we’ll have to satisfy
\[
\begin{align*}
0 & \leq \frac{\beta(x)}{U_2} + \Delta_x (u + \Delta_u) + \Delta_u x < 1.
\end{align*}
\]
The “problematic” term from (A6) is \( \Delta_x (u + \Delta_u) + \Delta_u x \) since \( \beta(x)/U_2 \) is always within a well defined range. Inputs \( \Delta_u x \) are not related to parameters \( u \) and \( \Delta_u \) so to prohibit growing of \( \Delta_x (u + \Delta_u) + \Delta_u x \) beyond all limits, we’ll have to choose \( \Delta_u = 0 \). That is, having \( u = u' \). (A6) has then the following form
\[
\begin{align*}
\left[ \frac{\beta(x)}{U_2} + \Delta_x \frac{U_1}{U_2} \right] & = 0 \tag{A7}
\end{align*}
\]
which is the basic equation for equivalence between the ILS and its RT counterpart. To satisfy (A7) we have to find minimum and maximum of
\[
\begin{align*}
v & = \left[ \frac{\beta(x)}{U_2} + \Delta_x \frac{U_1}{U_2} \right].
\end{align*}
\]
This can be split in two cases: $U_1 \geq 0$ and for $U_1 < 0$.

For $U_1 \geq 0$ we have

$$\min\{v\} = \beta(x) = 0 \quad \text{and} \quad \Delta_x = \frac{P_2 - 1}{P_2}$$

and

$$\max\{v\} = \frac{U_2 - 1}{U_2} + \frac{U_1 (P_2 - 1)}{U_2 P_2} \quad (\beta(x) = U_2 - 1, \quad \Delta_x = \frac{P_2 - 1}{P_2})$$

To satisfy (A7) we have to fulfill

$$0 \leq \frac{U_2 - 1}{U_2} + \frac{U_1 (P_2 - 1)}{U_2 P_2} < 1.$$ 

For $U_1 < 0$ we have

$$\min\{v\} = \frac{U_2 - 1}{U_2} \quad (\beta(x) = 0 \quad \text{and} \quad \Delta_x = \frac{P_2 - 1}{P_2});$$

and

$$\max\{v\} = \frac{U_2 - 1}{U_2} = 0 \quad (\Delta_x = 0 \quad \text{and} \quad \beta(x) = U_2 - 1).$$

To satisfy (A7) in this case we’ll have to fulfill

$$0 \leq \frac{U_2 - 1}{U_2} \quad \text{which obviously is} \quad P_2 = 1 \quad \text{since} \quad U_1 < 0.$$ 

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References


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