On the Relationship between Integer Lifting and Rounding Transform

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Abstract. *In this paper we analyze the relationship between integer Lifting scheme and Rounding transform as means to compute the wavelet transform in signal processing area. We bring some new results which better describe relationship, reversibility and equivalence of integer lifting scheme and rounding transform concept.*

Keywords

Wavelet, lifting, transform, integer, rounding.

1. Introduction

At present time, there is still an effort to introduce a new general concept of an integer-to-integer transform. The most known is the integer version of the lifting scheme [1], [2], [3] (see Fig. 1.) - the integer lifting scheme (ILS) [1], [2], [3] (see Fig. 2.). Lifting factorizes wavelet transform into successive invertible steps.

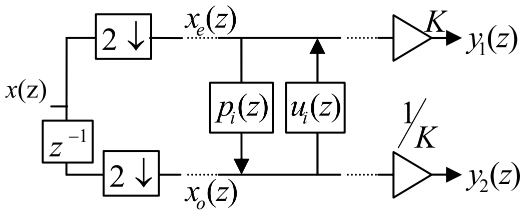

Fig. 1. Lifting scheme.

Applying rounding operators after each step, the ILS can be obtained [9].

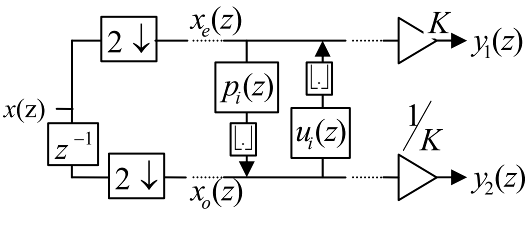

Fig. 2. Integer lifting scheme.

Besides the ILS, there is also a variety of transforms that operate as integer-to-integer transforms [10 - 13]. Although some of them are ad-hoc solutions, some have more generic construction. We've more closely examined the idea of Prost and Jung. Their concept is based on the floor operator applied after the forward transform and the ceil operator applied after the backward transform. They refer to this concept as the Rounding transform (RT) [4], [5], [7], [8]. The forward transform of RT is defined as following:

$$\mathbf{y} = \lfloor \mathbf{Hx} \rfloor . \tag{1}$$

The backward transform is defined:

$$\mathbf{x} = \lceil \mathbf{Vy} \rceil = \lceil \mathbf{V} \lfloor \mathbf{Hx} \rfloor \rceil \tag{2}$$

where matrix **V** is defined as inverse matrix to **H**. Later they introduced the overlapped Rounding transform (ORT) [6], where terms in (1) and (2) contained polynomials. Despite of the interesting idea, the authors were unable to show general solution that would satisfy (2). Instead they presented several types of matrices that satisfy the mentioned equation. Later, these particular matrices have been revisited by Adams and Kossentini [14] and have been found lifting equivalent.

We have examined the relationship between integer lifting and its rounding counter part (Fig. 3). First we show the set of equivalence, i.e. set where RT and ILS produce identical outputs for identical inputs with respect to *p ,u, p', u'*. This is shown in section 2. Then in section 3 we show, when the rounding transform is reversible and how important the relationship between the two approaches is.

2. Relationship between Integer Lifting and Rounding Transform

To show the relationship between ILS and RT, we've chosen a version of the ILS with one predict and update step (without the normalization step). This scheme is compared against its RT counterpart (rounding operation applied at the end of the forward transform) as it is shown in Fig. 3.

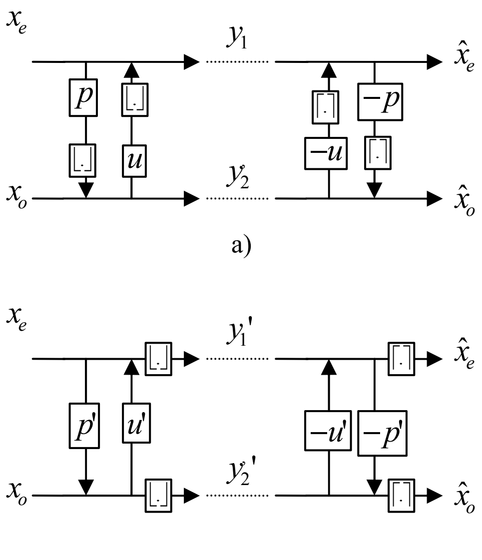

Fig. 3. Forward and backward part of a) Integer lifting, b) Rounding transform

Altough RT shown here is not a general RT (the **H** matrix for it is not a general one as we'll show later), it is a sufficient example to demonstrate how RT and ILS are related. Because both transforms are integer-to-integer transforms and the result should be represent able in a machine with finite precision, we will assume

$$x_e, x_o \in Z \text{ and } p, u, p', u' \in Q. \tag{3}$$

In appendix A it is shown, that there is no case, where both approaches would produce identical outputs for all possible (and identical) inputs, when $p \neq p'$ and $u \neq u'$ Then according to Fig. 3. the outputs for the ILS are

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_e + \lfloor u\{\lfloor px_e \rfloor + x_o\}\rfloor \\ \lfloor px_e \rfloor + x_o \end{pmatrix} \tag{4}$$

and the outputs produced by the RT scheme are

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_e + \lfloor u\{px_e + x_o\}\rfloor \\ \lfloor px_e \rfloor + x_o \end{pmatrix}. \tag{5}$$

From Fig. 3., it is clear, that $y_2 = y_2'$. To analyze the equivalence of ILS and RT, we'll need to find all p, u satisfying the equation

$$y_1 = y_1' \tag{6}$$

for all possible x_e, x_o. The equation to solve is:

$$x_e + \lfloor u\{\lfloor px_e \rfloor + x_o\}\rfloor = x_e + \lfloor u\{px_e + x_o\}\rfloor. \tag{7}$$

According to (3), we can express u and p in the following form

$$u = \frac{U_1}{U_2},\ p = \frac{P_1}{P_2} \text{ where}$$

$$U_1, P_1 \in Z \wedge U_2, P_2 \in N - \{0\}. \tag{8}$$

Equation (7) is satisfied only when

$P_1 = 1$ for $U_1 < 0$ or

$$0 \leq \frac{U_2 - 1}{U_2} + \frac{U_1}{U_2}\frac{(P_2 - 1)}{P_2} < 1 \text{ for } U_1 \geq 0. \tag{9}$$

We call the set of p and u, where (9) holds, the set of equivalence. It is depicted in Fig. 4.

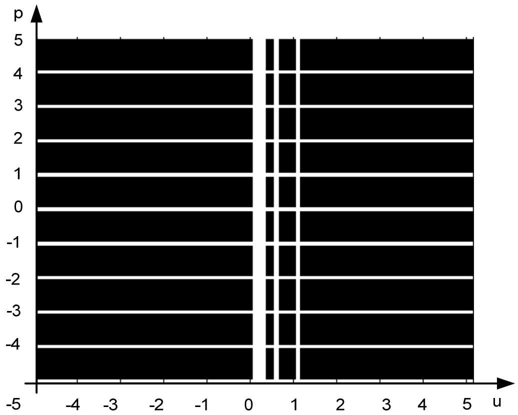

Fig. 4. Set of equivalence of ILS and RT. White area depicts where the equation (9) holds and black area where it doesn't. Displayed is range $p,u \in$<-5, 5>.

3. Reversibility of Rounding Transform

To examine the reversibility of RT, we'll have to examine the whole scheme, not only the forward part of it (as in section 2). The scheme in the Fig. 3b corresponds to (2) with the following forward and backward matrix:

$$\mathbf{H} = \begin{pmatrix} 1+up & u \\ p & 1 \end{pmatrix},\quad \mathbf{V} = \mathbf{H}^{-1} = \begin{pmatrix} 1 & -u \\ -p & 1+up \end{pmatrix}. \tag{10}$$

Then for outputs $\hat{x}_e$ and $\hat{x}_o$ it holds:

$$\begin{pmatrix} \hat{x}_e \\ \hat{x}_o \end{pmatrix} = \left\lceil \begin{pmatrix} x_e + \lfloor upx_e + ux_o \rfloor - u\lfloor px_e \rfloor - ux_o \\ x_o + \lfloor px_e \rfloor + up\lfloor px_e + x_o \rfloor - px_e - p\lfloor u(px_e + x_o)\rfloor \end{pmatrix} \right\rceil. \tag{11}$$

The reversibility is achieved when $\hat{x}_e = x_e$ and $\hat{x}_o = x_o$. Solution of (11) is not trivial, but a closer look on the reconstruction of x_e in ILS and RT approach reveals that it is done in the same way. Based on Fig. 3. for lifting

$$x_e = y_1 + \lceil -y_2 u \rceil = y_1 - \lfloor y_2 u \rfloor \tag{12}$$

and rounding

$$x_e = y'_1 - \lfloor y_2 u \rfloor. \tag{13}$$

Equation (12) always holds, since the integer lifting is reversible. Following this, (13) can hold only if $y_1 = y'_1$. Considering $y_2 = y'_2$ this finally tells us: if the RT is reversible, then it is possible to write it in an ILS form.

4. Conclusion

From what we have shown in section 3, one can see that the forward part of RT is in fact ILS in our case. Nothing is said about the reconstruction of $\hat{x}_o$. Solving $\hat{x}_o = x_o$ from (11) is difficult, since the backward part of RT is different than in ILS. Following results that we've derived in section 3, it's obvious that the set of reversibility will be not greater than the set of equivalence. Expansion of this statement to a polynomial case should be straightforward.

Appendix A - Equivalence of ILS and RT Approach

The output of ILS, according to Fig. 3. is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_e + \lfloor u\{\lfloor px_e \rfloor + x_o\}\rfloor \\ \lfloor px_e \rfloor + x_o \end{pmatrix}$$

and for RT we have

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_e + \lfloor u'\{p'x_e + x_o\}\rfloor \\ \lfloor p'x_e \rfloor + x_o \end{pmatrix}.$$

The equivalence is achieved when

$y_2 = y'_2$, i. e. $\lfloor px_e \rfloor + x_o = \lfloor p'x_e \rfloor + x_o$ (A1)

$y_1 = y'_1$, i. e. $x_e + \lfloor u\{\lfloor px_e \rfloor + x_o\}\rfloor = x_e + \lfloor u'\{p'x_e + x_o\}\rfloor$ (A2)

where $p, u, p', u' \in Q$ (since working with finite precision). In the following we'll use these properties:

1. $a = \frac{A_1}{A_2}$ where $A_1 \in Z \wedge A_2 \in N - \{0\}$,
2. for every Ax where $A \in Q$, $x \in Z$ we can write $Ax = \frac{A_1}{A_2}x = C(x) + \frac{D(x)}{A_2}$ where $D(x) \in \left\{0,1,\ldots \frac{A_2 - 1}{A_2}\right\}$ and $A_1 \in Z$ $A_2 \in N - \{0\}$

Solution (A1)

We can rewrite p' as $p' = p + \Delta_p$ so we obtain $\lfloor px_e \rfloor \overset{!}{=} \lfloor px_e + \Delta_p x_e \rfloor$. It's clear that the right term of (A1) will grow over all limits with nonzero Δ_p. The equation (A1) would not hold for all x_e in this case. Therefore, to fulfill (A1) we have to choose: $\Delta_p = 0$, which means $p' = p$.

Solution (A2)

Instead of using $px_e + x_0$, we'll express it as $x = px_e + x_0 = (P_1/P_2)x_e + x_o$. This shows that x is either integer or an integer plus a fraction of P_2. This way, x can be expressed also as $x = \lfloor x \rfloor + \Delta_x$ whereby $\Delta_x \in \{0,1,\ldots,(P_2-1)/P_2\}$. If we write $u' = u + \Delta_u$ we can express (A2) as follows

$$\lfloor u\lfloor x \rfloor\rfloor = \lfloor u\lfloor x \rfloor + u\Delta_x + \Delta_u x \rfloor. \tag{A3}$$

$u\lfloor x \rfloor$ can be expressed as

$$u\lfloor x \rfloor = \frac{U_1}{U_2}\lfloor x \rfloor = \alpha(x) + \frac{\beta(x)}{U_2}$$

whereby $\beta(x) \in \left\{0,1,\ldots,\frac{U_2 - 1}{U_2}\right\}$. The (A3) changes then to

$$\alpha(x) + \left\lfloor \frac{\beta(x)}{U_2} \right\rfloor = \alpha(x) + \left\lfloor \frac{\beta(x)}{U_2} + u\Delta_x + \Delta_u x \right\rfloor \tag{A4}$$

and since $\left\lfloor \frac{\beta(x)}{U_2} \right\rfloor = 0$ further

$$0 = \left\lfloor \frac{\beta(x)}{U_2} + u\Delta_x + \Delta_u x \right\rfloor. \tag{A5}$$

Substituting one more time $x = \lfloor x \rfloor + \Delta_x$ we obtain

$$\left\lfloor \frac{\beta(x)}{U_2} + \Delta_x(u + \Delta_u) + \Delta_u \lfloor x \rfloor \right\rfloor \overset{!}{=} 0. \tag{A6}$$

That means, that we'll have to satisfy

$$0 \le \frac{\beta(x)}{U_2} + \Delta_x(u + \Delta_u) + \Delta_u \lfloor x \rfloor < 1.$$

The "problematic" term from (A6) is $\Delta_x(u + \Delta_u) + \Delta_u \lfloor x \rfloor$, since $\beta(x)/U_2$ is always within a well defined range. Inputs Δ_x, $\lfloor x \rfloor$ are not related to parameters u and Δ_u, so to prohibit growing of $\Delta_x(u + \Delta_u) + \Delta_u \lfloor x \rfloor$ beyond all limits, we'll have to choose $\Delta_u = 0$. That is, having $u = u'$. (A6) has then the following form

$$\left\lfloor \frac{\beta(x)}{U_2} + \Delta_x \frac{U_1}{U_2} \right\rfloor \overset{!}{=} 0 \tag{A7}$$

which is the basic equation for equivalence between the ILS and its RT counterpart. To satisfy (A7) we have to find minimum and maximum of

$$v = \left\lfloor \frac{\beta(x)}{U_2} + \Delta_x \frac{U_1}{U_2} \right\rfloor.$$

This can be split in two cases: $U_1 \geq 0$ and for $U_1 < 0$.

For $U_1 \geq 0$ we have

$\min\{v\}$ equal to zero ($\beta(x) = 0$ and $\Delta_x = 0$) and $\max\{v\} = \left\lfloor \frac{U_2 - 1}{U_2} + \frac{U_1}{U_2}\frac{(P_2 - 1)}{P_2} \right\rfloor$ ($\beta(x) = U_2 - 1$, $\Delta_x = \frac{P_2 - 1}{P_2}$).

To satisfy (A7) we have to fulfill

$$0 \leq \frac{U_2 - 1}{U_2} + \frac{U_1}{U_2}\frac{(P_2 - 1)}{P_2} < 1 .$$

For $U_1 < 0$ we have

$\min\{v\} = \left\lfloor \frac{U_1}{U_2} . \frac{(P_2 - 1)}{P_2} \right\rfloor$ ($\beta(x) = 0$ and $\Delta_x = \frac{P_2 - 1}{P_2}$);

$\max\{v\} = \left\lfloor \frac{U_2 - 1}{U_2} \right\rfloor = 0$ ($\Delta_x = 0$ and $\beta(x) = U_2 - 1$).

To satisfy (A7) in this case we'll have to fulfill

$$0 \leq \frac{U_1}{U_2} . \frac{(P_2 - 1)}{P_2} < 1$$

which obviously is $P_2 = 1$ since $U_1 < 0$.

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References

[1] BURRUS, S., GOPINATH, R., GUO, H. *Introduction to Wavelets and Wavelet Transforms.* Prentice Hall, 1998, ISBN 0-13-489600-9.

[2] VETTERLI, M., KOVACEVIC, J. *Wavelets and Subband Coding*. Prentice Hall, 1995, ISBN 0-13-097080-8.

[3] STRANG, G., NGUYEN, T. *Wavelets and Filter Banks*. Wellesley-Cambridge Press, 1996, ISBN 0-9614088-7-1.

[4] JUNG, H., PROST, R. Rounding transform and its application for lossless pyramid structured coding. *IEEE Transactions on Image Processing*, 1998, vol. 7, no. 2.

[5] JUNG, H., PROST, R. Rounding transform based approach for lossless subband coding. In *International Conference on Image Processing*, 1997.

[6] JUNG, H., PROST, R. Lossless subband coding system based on rounding transform. *IEEE Transactions on Signal Processing*, 1998, vol. 46, no. 9.

[7] JUNG, H., PROST, R. Multi-port filtering system for lossless image compression. In *International Conference on Image Processing,* 1998.

[8] JUNG, H., PROST, R. Multi-channel overlapping rounding transform for HINT and subband lossless image compression. *Signal Processing*, 2000, vol. 80, no. 12.

[9] CALDERBANK, R., DAUBECHIES, I. Wavelet transforms that map integers to integers. *Applied and Computational Harmonic Analysis (ACHA)*, 1998, vol. 5, no. 3.

[10] CHAO, P. An approach to fast integer wavelet transforms for lossless for image compression. In *Proc. of International Symposium on Computational Mathematics*. Guangzhou (China), 1997.

[11] PROCHÁSKA, J., VARGIC, R. Compression of bitonal images using nonlinear wavelet transform. In *Proc. 6th International Conference Research in Telecommunication Technology RTT 2005*. Hradec nad Moravicí (Czech Republic), Sept. 12-14, 2005, p. 587-591.

[12] ZANDI, A., ALLEN, J. D. CREW: Compression with Reversible Embedded Wavelets. In *Proceedings of the Conference on Data Compression*, 1995.

[13] SAID, A., PEARLMAN, W. A. An image multiresolution representation for lossless and lossy compression. *IEEE Transactions on Image Processing*, 1996, vol. 5, no. 9, p.1303 – 1310.

[14] ADAMS, M., KOSSENTINI F. On the relationship between the overlapping rounding transform and lifting frameworks for reversible subband transforms. *IEEE Transactions on Signal Processing*, 2000, vol. 48, no. 1, p. 261-266.

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