On the Modal Superposition Lying under the MoM Matrix Equations

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Abstract. *This paper shows an interesting sight on the correspondence between total current solution obtained by the Method of Moments and by the Characteristic modes respectively. Derivation is based on representation of a matrix in its spectral form. Detailed study of dipole antenna fed under different conditions is presented demonstrating that modal decomposition of currents offers a nice physical insight into behavior of radiating structures.*

Keywords

method of moments, characteristic modes, matrix operator, spectral factorization, modal superposition.

1. Introduction

Solving current distribution on metal surfaces usually incorporates formulation of the EFIE (Electric Field Integral Equation) [1]. Initial operator equation can be formulated by employing boundary condition for tangential incident (E^{i}) and scattered (E^{s}) electric field[1] $E^{\mathrm{i}} + E^{\mathrm{s}} = 0$:

$$\left[L(J) - E^{\mathrm{i}}\right]_{\tan} = 0. \tag{1}$$

The operator used in (1) is defined by

$$L(J) = j\omega A(J) + \nabla\phi(J) \tag{2}$$

where $A(J)$ and $\phi(J)$ are vector and scalar potentials respectively, see e.g. [1] for their definition.

Physically, $-L(J)$ gives the scattered electric field intensity E^s at any point of space due to current J on surface S. Therefore, the operator L has the dimension of impedance:

$$Z(J) = \left[L(J)\right]_{\tan}. \tag{3}$$

This impedance operator $Z = R + jX$ is a symmetric but not Hermitian operator, but since Z is symmetric, its Hermitian parts R and X are real symmetric operators. Solution of (1) is usually treated by the mathematic procedure called Method of Moments.

[1] we will omit vector nature of the fields for easier notation

The total current distribution is then obtained by **Z** matrix inversion (direct method), [1], [2]. An alternative approach is modal superposition of the **Z** matrix via Characteristic modes, [6], [8]. In the following we will show that both solution methods are equivalent.

2. Method of Moments

Consider general operator equation

$$L(g) = h \tag{4}$$

where the linear operator L is applied to an unknown function g to be found and h is a known excitation function. The MoM solution procedure for (4) is well-known, [1], [2]. In fact, the MoM procedure turns continuous operator equation (4) into its discrete approximation:

$$\sum_{n=1}^{N} a_n \langle w_m, L(g_n)\rangle = \langle w_m, h\rangle, \;\; m = 1, 2, \ldots N \tag{5}$$

where a_n are expansion coefficients, g_n basis (expansion) functions and w_n testing functions. Eq. (5) may be effectively written in a matrix form:

$$\mathbf{La} = \mathbf{h}, \;\; \mathbf{L} = [L_{mn}], \mathbf{a} = [a_n], \mathbf{h} = [h_m]. \tag{6}$$

Now the unknown expansion coefficients a_n in (5) are obtained by matrix inversion of **L**:

$$\mathbf{a} = \mathbf{L}^{-1}\mathbf{h}. \tag{7}$$

Let's turn now into the EFIE version of (4) for unknown surface currents on structure J

$$L(J) = E^{\mathrm{i}} \tag{8}$$

where L is defined by (2) and E^{i} is a known incident electric field intensity (i.e. described as a gap source of linear dipole or incident plane wave, see later). Applying the MoM procedure, one obtain the well-known matrix system

$$\mathbf{ZJ} = \mathbf{E}^{\mathrm{i}}, \;\; \mathbf{Z} = [Z_{mn}], \mathbf{J} = [J_n], \mathbf{E}^{\mathrm{i}} = [E_m]^{\mathrm{i}}. \tag{9}$$

Note that operator $L(J)$ has physical meaning of impedance, (3), thus $[L_{mn}]$=$[Z_{mn}]$ being the complex MoM

$\mathbf{Z} = \mathbf{R} + j\mathbf{X}$ matrix. For Galerkin's case, $\mathbf{Z}$ is $N \times N$ symmetric matrix. The vector of total surface currents $\mathbf{J}$ is obtained by inversion of $\mathbf{Z}$ (direct solution):

$$\mathbf{J} = \mathbf{Z}^{-1}\mathbf{E}^{\mathrm{i}}. \tag{10}$$

3. Theory of Characteristic Modes

Theory of Characteristic modes (TCM) evaluates total surface currents as a sum over so-called characteristic currents (eigencurrents) which are independent of any excitation. A reader is referred to [3], [6], [8], [9] for detailed information. Having the complex $\mathbf{Z} = \mathbf{R} + j\mathbf{X}$ impedance matrix, we consider the following weighted eigenvalue equation [4]:

$$\mathbf{ZJ} = \nu\mathbf{MJ} \tag{11}$$

where $\mathbf{M}$ is the weight operator which is chosen $\mathbf{M} = \mathbf{R}$ to give orthogonality of the radiation patterns. We obtain:

$$(\mathbf{R} + j\mathbf{X})\mathbf{J} = \nu\mathbf{RJ}. \tag{12}$$

Now next let

$$\nu = 1 + j\lambda \tag{13}$$

and after a little manipulation we get

$$\mathbf{XJ} = \lambda\mathbf{RJ}. \tag{14}$$

A Solution of the problem (14) may be easily obtained numerically, i.e. using `eig` in MATLAB. The modal decomposition (14) of the square impedance matrix of order N ($N \times N$) produces N eigencurrents $\mathbf{J}_n$ with N corresponding eigenvalues λ_n. Such modes form a complete set of solution, and hence the total current of a conducting body can be expressed as a linear superposition of these mode currents [6] (at a given frequency for which the $\mathbf{Z}$ is calculated):

$$\mathbf{J} = \sum_{n=1}^{N} b_n\mathbf{J}_n = \sum_{n=1}^{N} \frac{\langle \mathbf{J}_n, \mathbf{E}^{\mathrm{i}} \rangle}{1 + j\lambda_n}\mathbf{J}_n \tag{15}$$

where the coefficients b_n (modal amplitudes) are given by:

$$b_n = \frac{\langle \mathbf{J}_n, \mathbf{E}^{\mathrm{i}} \rangle}{1 + j\lambda_n} = \frac{\mathbf{J}_n^{\mathrm{T}}\mathbf{E}^{\mathrm{i}}}{1 + j\lambda_n} \tag{16}$$

where $\mathbf{J}_n^{\mathrm{T}}\mathbf{E}^{\mathrm{i}}$ is called the *modal excitation coefficient (factor)*. For both MoM and TCM, the excitation vector $\mathbf{E}^{\mathrm{i}}$ has the same meaning. For example, normal illumination of a structure by a plane wave will require $\mathbf{E}^{\mathrm{i}} = \{1\}_1^{\mathrm{N}}$. Eq. (15) resembles a solution using Green's function [1] (see later).

4. Modal Superposition

Have a look at (10) and (15) for the total current obtained by MoM matrix inversion and by the TCM eigencurrents superposition. Since both these equations refer to the same total current density $\mathbf{J}$, we obtain an interesting relation:

$$\mathbf{J} = \mathbf{Z}^{-1}\mathbf{E}^{\mathrm{i}} = \sum_{n=1}^{N} \frac{\mathbf{J}_n^{\mathrm{T}}\mathbf{E}^{\mathrm{i}}}{1 + j\lambda_n}\mathbf{J}_n. \tag{17}$$

This implies that MoM equation for total current $\mathbf{J} = \mathbf{Z}^{-1}\mathbf{E}^{\mathrm{i}}$ is in fact modal superposition of the characteristic currents, i.e. the matrix operation $\mathbf{Z}^{-1}$ conceals deep eigenmode superposition. To confirm that, we need to introduce some algebra.

4.1 Eigenvalue Decomposition

Eigenvalue decomposition is a special case of the Singular Value Decomposition (SVD), [7] for square $N \times N$ matrices. Such *spectral decomposition* (factorization) of the matrix $\mathbf{A}$ is defined as:

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{\mathrm{T}}. \tag{18}$$

The column vectors of $\mathbf{P}$ are the eigenvectors $\mathbf{u}_n$ of $\mathbf{A}$ and they are orthonormal. $\mathbf{\Lambda}$ is the diagonal matrix with λ_n at the diagonal:

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_N \end{pmatrix}. \tag{19}$$

Then the matrix $\mathbf{A}$ may be expressed in a *spectral form* as superposition over its eigenvectors

$$\mathbf{A} = \lambda_1\mathbf{u}_1\mathbf{u}_1^{\mathrm{T}} + \ldots + \lambda_N\mathbf{u}_N\mathbf{u}_N^{\mathrm{T}} = \sum_{n=1}^{N} \lambda_n\mathbf{u}_n\mathbf{u}_n^{\mathrm{T}}. \tag{20}$$

where $\mathbf{u}_n$ are eigenvectors and λ_n eigenvalues of $\mathbf{A}$ fulfilling

$$\mathbf{Au} = \lambda\mathbf{u}. \tag{21}$$

Using (20), matrix inversion of $\mathbf{A}$ is consequently defined as:

$$\mathbf{A}^{-1} = \mathbf{P}\mathbf{\Lambda}^{-1}\mathbf{P} = \sum_{n=1}^{N} \frac{1}{\lambda_n}\mathbf{u}_n\mathbf{u}_n^{\mathrm{T}}. \tag{22}$$

The inverse matrix operator $\mathbf{A}^{-1}$ is expressed as eigenvector superposition; that's the key to the original problem (17).

5. The MoM Matrix Equation

Let's go back to the MoM equation for total current

$$\mathbf{J} = \mathbf{Z}^{-1}\mathbf{E}^{\mathrm{i}} \tag{23}$$

where $\mathbf{Z} = \mathbf{R} + j\mathbf{X}$, $\mathbf{R}$ and $\mathbf{X}$ being real symmetric matrix operators of order N. Applying (14), the spectral formulation for $\mathbf{Z}^{-1}$ is:

$$\mathbf{Z}^{-1} = \sum_{n=1}^{N} \frac{\mathbf{J}_n\mathbf{J}_n^{\mathrm{T}}}{\nu} = \sum_{n=1}^{N} \frac{\mathbf{J}_n\mathbf{J}_n^{\mathrm{T}}}{1 + j\lambda_n} \tag{24}$$

and consequently

$$\mathbf{Z}^{-1}\mathbf{E} = \sum_{n=1}^{N} \frac{\mathbf{J}_n \mathbf{J}_n^{\mathrm{T}}}{1 + j\lambda_n} \mathbf{E}^{\mathrm{i}} = \sum_{n=1}^{N} \frac{\mathbf{J}_n^{\mathrm{T}} \mathbf{E}^{\mathrm{i}}}{1 + j\lambda_n} \mathbf{J}_n. \qquad (25)$$

Eq. (25) presents a clear physical view on the modal superposition hidden in the direct solution (23). Actually, the term $\mathbf{Z}^{-1}\mathbf{E}^{\mathrm{i}}$ acts as a *superposition of characteristic modes* weighted by *modal excitation factor*, see also Fig.1. To the best author's knowledge, this interesting relation hasn't been pointed out in a recent literature ([8], [3], [9] to cite a few). Furthermore, it has to be noted that the operator $\mathbf{Z}^{-1}$ may be understood as a discrete *Green's function*. A short remark (but with no derivation of it) on the spectral form of $\mathbf{Z}^{-1}$, eq. (24), is mentioned in [6].

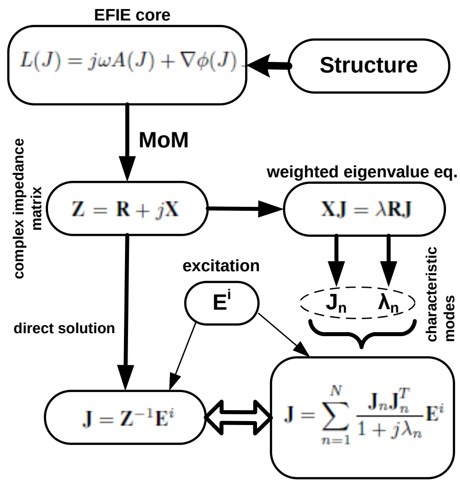

Fig. 1. Solution for total current $\mathbf{J}$, correspondence between direct MoM solution and modal superposition.

6. Example

Consider a thin dipole of overall length $2L$ and radius a, excited under different conditions: by a plane wave and by a center gap feed, see Fig.2. Simple thin-wire-kernel ($2L/a = 2000$) MoM code has been used, with $N = 61$ segments. Simulation parameters are $f = 300$ MHz ($\lambda = 1$ m) and $2L = 2$ m.

At first, 9 characteristic currents J_n have been calculated together with their associated eigenvalues λ_n. The modes have to be normalized to fulfill that each eigencurrent radiates a unit power [8]:

$$\langle \mathbf{J}_n, R\mathbf{J}_n \rangle = 1, \qquad (26)$$

so the normalized eigencurrents $\mathbf{j}_n$ are obtained as follows

$$\mathbf{j}_n = \frac{\mathbf{J}_n}{\sqrt{\langle \mathbf{J}_n, R\mathbf{J}_n \rangle}} = \frac{\mathbf{J}_n}{\sqrt{\mathbf{J}_n^{\mathrm{T}} R \mathbf{J}_n}}. \qquad (27)$$

First 4 normalized currents[2] are plotted at Fig. 3.

[2]as eigencurrent shapes vary with frequency [8], out-of-resonance modes needn't to always fulfill zero current at the dipole ends.

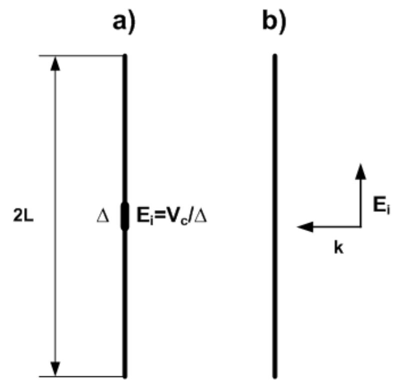

Fig. 2. Dipole excited with center voltage gap (a) and by a plane wave (b).

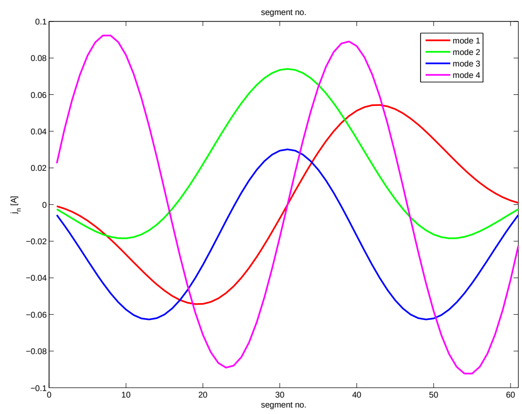

Fig. 3. First 4 normalized characteristic current $\mathbf{j}_1$ - $\mathbf{j}_4$ along the $2L = 2$ m dipole @ 300 MHz.

According to (15), the contribution to the total current can be decomposed into the following parts:

- excitation coefficient, that is inner product of $\langle \mathbf{J}_n, \mathbf{E}^{\mathrm{i}} \rangle$ which models coupling of modes with their excitation
- modal amplitude $\left| \frac{1}{1+j\lambda_n} \right|$
- amplitude of eigencurrents normalized to unit radiated power, see (27)

For the given dipole, mode 4 has the smallest eigenvalue, close to 0 (that indicates resonance, [8]). Its current pattern (Fig.3) has two wavelengths thus one can say that this dipole is working with $2L = 2\lambda$ resonance mode. This can be easily confirmed recalling that $2L = 2$ m and $\lambda = 3 \cdot 10^8 / 300 \cdot 10^6 = 1$ m. However, as it was described above, the eigenvalues themselves are not sufficient enough to say that some mode will be dominant in the total current distribution, but one has to consider the product of coupling coefficient and modal amplitude, i.e. the value $\left| \frac{\langle \mathbf{J}_n, \mathbf{E}^{\mathrm{i}} \rangle}{1+j\lambda_n} \right|$.

6.1 Plane Wave Illumination

We will study now the current distribution along our

$2L = 2\lambda$ dipole in ”receiving mode”, i.e. as illuminated by normally incident plane wave $E_i = 1$ V/m, Fig.2 b). Excitation vector $\mathbf{E}^{\mathrm{i}}$ in (15) will thus have form $\mathbf{E}^{\mathrm{i}} = [E^{\mathrm{i}}, E^{\mathrm{i}}, E^{\mathrm{i}}...]^{\mathrm{T}}$. Superposition coefficients playing role in (15) are shown in Tab. 1.

n	λ_n	$\langle \mathbf{J}_n, \mathbf{E}^{\mathrm{i}} \rangle$	$\left\lvert \frac{1}{1+j\lambda_n} \right\rvert$	$\left\lvert \frac{\langle \mathbf{J}_n, \mathbf{E}^{\mathrm{i}} \rangle}{1+j\lambda_n} \right\rvert$
1	3.39	$-5.6 \cdot 10^{-13}$	0.283	$-1.6 \cdot 10^{-13}$
2	3.35	0.894	0.286	0.256
3	3.24	-1.565	0.295	-0.46
4	0.49	$3.1 \cdot 10^{-15}$	0.898	$2.7 \cdot 10^{-15}$
5	-36.9	0.91	0.027	0.025
6	-616	$-1.37 \cdot 10^{-15}$	0.016	$-2.2 \cdot 10^{-18}$
7	$-1.2 \cdot 10^{4}$	-0.94	$8.2 \cdot 10^{-5}$	$-7.6 \cdot 10^{-5}$
8	$-3.1 \cdot 10^{7}$	$2.5 \cdot 10^{-13}$	$3.2 \cdot 10^{-6}$	$8.0 \cdot 10^{-19}$
9	$-1.0 \cdot 10^{7}$	-0.96	$9.8 \cdot 10^{-8}$	$-9.4 \cdot 10^{-8}$

Tab. 1. Superposition coefficients for a dipole excited by a plane wave.

Fig. 4 shows the real, imaginary and absolute value of a current flowing along the dipole as calculated by superposition of different modes up to count of 9. According to Tab.1, only modes 2, 3 and 5 are significant (note that mode 4 with the lowest eigenvalue almost doesn't contribute). It can be seen that convergence is achieved after summing up relatively low number of modes ($\sim$5).

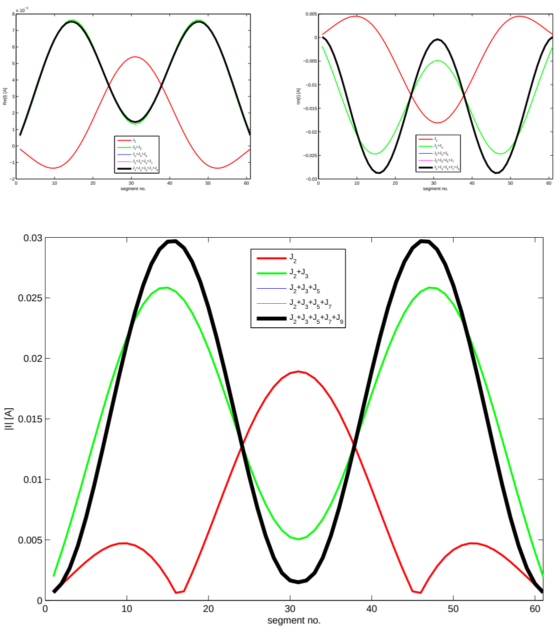

Fig. 4. $\mathrm{Re}\{I\}$ (top left), $\mathrm{Im}\{I\}$ (top right) and $|I|$ (bottom) for 2λ dipole excited by a plane wave.

6.2 Center Segment Excitation

The second case is introduced by $2L = 2\lambda$ dipole in ”transmitting mode”, i.e. excited at the center segment by a voltage delta gap [1], see Fig.2 a). Excitation vector $\mathbf{E}^{\mathrm{i}}$ in (15) will thus have form $\mathbf{E}^{\mathrm{i}} = [0, 0, ..., V_c, ...0, 0]^T$, where $V_c = \frac{E_i}{\Delta} = \frac{E_i}{2L/N}$ and $E_{\mathrm{i}} = 1$ V/m has been chosen. Again, superposition coefficients playing role in (15) are presented - see Tab. 2 and note of the mode 3 sign change compared to the previous case. Fig. 5 shows how the total current along

n	λ_n	$\langle \mathbf{J}_n, \mathbf{E}^{\mathrm{i}} \rangle$	$\left\lvert \frac{1}{1+j\lambda_n} \right\rvert$	$\left\lvert \frac{\langle \mathbf{J}_n, \mathbf{E}^{\mathrm{i}} \rangle}{1+j\lambda_n} \right\rvert$
1	3.39	$-1.6 \cdot 10^{-12}$	0.283	$-4.6 \cdot 10^{-13}$
2	3.35	2.258	0.286	0.655
3	3.24	0.919	0.295	0.271
4	0.49	$1.1 \cdot 10^{-13}$	0.898	$9.8 \cdot 10^{-14}$
5	-36.9	5.753	0.027	0.156
6	-616	$5 \cdot 10^{-16}$	0.016	$8.1 \cdot 10^{-19}$
7	$-1.2 \cdot 10^{4}$	59.7	$8.2 \cdot 10^{-5}$	$5 \cdot 10^{-3}$
8	$-3.1 \cdot 10^{7}$	$-1.3 \cdot 10^{-11}$	$3.2 \cdot 10^{-6}$	$-4.0 \cdot 10^{-17}$
9	$-1.0 \cdot 10^{7}$	$-1.3 \cdot 10^{3}$	$9.8 \cdot 10^{-8}$	$-1.3 \cdot 10^{-4}$

Tab. 2. Superposition coefficients for a dipole excited at the center segment.

the dipole is formed when more modes are added. It could be seen that in this case the convergence is much worse for the imaginary part of current. This issue is known well but exceeds the aim of this paper, we refer to [8] for more details.

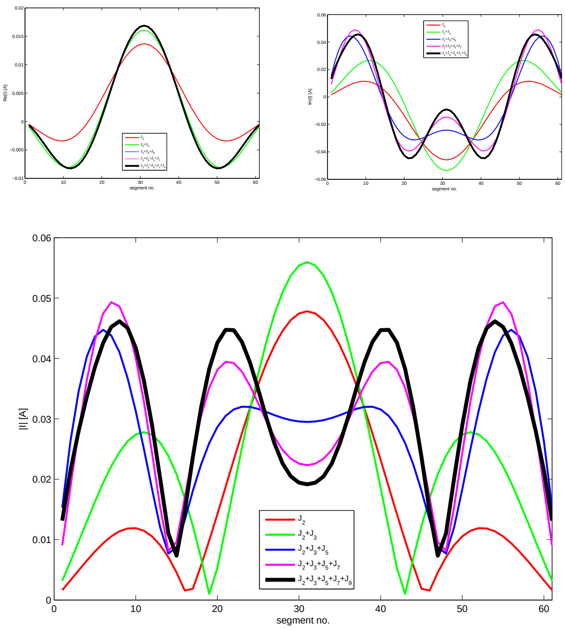

Fig. 5. $\mathrm{Re}\{I\}$ (top left), $\mathrm{Im}\{I\}$ (top right) and $|I|$ (bottom) for 2λ dipole excited at the center segment.

Current distributions obtained by the modal superposition confirms FEKO MoM code [10] used for comparison (Fig.6), only $|I|$ shown.

7. Conclusions

Using the singular value decomposition and later by its special case - spectral decomposition of a matrix - we

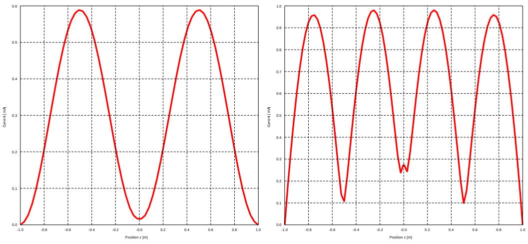

Fig. 6. FEKO results for the total current $|I|$ along the 2λ dipole: excitation by a plane wave (left) and at the center segment (right).

showed that the Method of Moments equation for the total current is equivalent to the superposition of characteristic modes. In other words, the inverse impedance matrix operator may be expressed in spectral form as a sum over eigenmodes of the related generalized eigenvalue equation. By calculating the total current by matrix inversion and feeding incorporated, we are generally loosing information on the underlying current modes. This is very similar to the collapse of states (i.e. modes) in quantum mechanics. An advantage of using the modal decomposition has been demonstrated by the example of the dipole excited in different ways. Unfortunately (and authors wonder why), no commercial software package developer using integral solver has incorporated this modal technique yet. Since the impedance matrix has to be calculated anyway for the direct solution, only very little computational effort is needed to perform further matrix modal decomposition. The reason for a such modal aspect is to gain precise physical insight on antenna behavior by "decoding" of total complex current density into a much more comprehensible basic "building blocks" - the modes with their eigenvalues.

Acknowledgement

This work was supported by the grant 102/08/H018 and the project COST ASSIST IC0603. Authors wish to acknowledge the help of Dr. Jaroslav Láčik, Dr. Veronika Sobotíková and Dr. Stanislav Zvánovec.

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