Optimal Piecewise-Linear Approximation of the Quadratic Chaotic Dynamics

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Abstract. *This paper shows the influence of piecewise-linear approximation on the global dynamics associated with autonomous third-order dynamical systems with the quadratic vector fields. The novel method for optimal nonlinear function approximation preserving the system behavior is proposed and experimentally verified. This approach is based on the calculation of the state attractor metric dimension inside a stochastic optimization routine. The approximated systems are compared to the original by means of the numerical integration.*

Real electronic circuits representing individual dynamical systems are derived using classical as well as integrator-based synthesis and verified by time-domain analysis in Orcad Pspice simulator. The universality of the proposed method is briefly discussed, especially from the viewpoint of the higher-order dynamical systems. Future topics and perspectives are also provided.

Keywords

Chaotic dynamics, Lyapunov exponents, piecewise-linear approximation, stochastic optimization.

1. Introduction

Recent studies reveal that chaos is not restricted only to very complex dynamical systems but can be observed also in the case of algebraically simple set of differential equations with six terms including nonlinearity [1], [2]. This is important because it has immediate contribution to the theory of lumped circuits with accumulating elements. None of them can be treated as linear without the loss of generality. Moreover, many widely used networks already exhibit irregular, noise-like motion called chaos [3], [4]. These fundamental circuits become inseparable parts of the large systems dedicated to analog signal processing. Chaos has been already reported in DC-DC converters [5], PLL circuits [6], biquads and universal filters [7], multiple logic cells [8], analog neural networks [9], etc. Discovering new circuit topologies which generate chaotic waveform for some internal parameters and initial conditions becomes the area of interest for many scientists. The straightforward procedure dealing with circuit synthesis for modeling nonlinear dynamics can be found in [10].

By definition it is not possible to obtain the closed-form analytic solution of the nonlinear dynamical systems with possible chaotic motion. The majority of the existing methods for analysis are based on the numerical integration process. This holds for calculation of the one-dimensional bifurcation diagrams [11], the basins of attraction [12] or spectrum of the one-dimensional Lyapunov exponents (LE) [13]. It turns out that the latter case can be utilized for the dynamical flow quantification, which is quickly calculated and accurate enough. These three real numbers (in the case of dynamical system with three degrees of freedom) uniquely determine the so-called Kaplan-Yorke metric dimension of the state space attractor (KYD) [14]

$$d_{\mathrm{KY}} = d_{\mathrm{T}} + \left| \ell_{d_{\mathrm{T}}+1} \right|^{-1} \sum_{i=1}^{d_{\mathrm{T}}} \ell_i \qquad (1)$$

where ℓ_i are individual LE sorted in descending order and d_{T} is a topological dimension. For dynamical systems under inspection holds $d_{\mathrm{T}} = 2$. The formula for LE is following

$$\ell(\mathbf{x}_0, \mathbf{y}_0 \in \mathbf{T} \cdot \mathbf{x}(\mathrm{t})) = \lim_{\mathrm{t}\to\infty} \frac{1}{\mathrm{t}} \frac{\| \mathrm{D_X}\Phi(\mathrm{t}, \mathbf{x}_0)\mathbf{y}_0 \|}{\| \mathbf{y}_0 \|} \qquad (2)$$

where $\mathbf{T}\cdot\mathbf{x}(\mathrm{t})$ denotes the tangent space in the fiducial point on the state trajectory and double brackets represent norm in Euclidean space. There are several other ways to determine the metric dimension of the state space attractor. According to the author's opinion only this one is suitable for practical calculations. Other possibilities are characterized by slow calculation process or inaccurate results for practical utilization. One such approach is based on the summarization of the cubes in the state space which are eventually occupied by the state space attractor under inspection. This kind of behavior quantifier is known under notion capacity dimension [15] and its final value can be roughly established as a slope of curve ln(N) vs ln(ε) before saturation, in detail

$$d_{\mathrm{CAP}} = \lim_{\varepsilon\to 0} (\ln N)/(\ln \varepsilon) \qquad (3)$$

where ε represents volume element edge and N is the total number of cubes necessary to fully cover entire attractor.

To end this discussion there are also routines for LE extraction from measured time series, i.e. in the case if original mathematical model is unknown. State attractor reconstruction is mostly based on the time gap approach [16] and its results are extremely sensitive to the choice of basic procedure parameters, i.e. time delay and embedding dimension. Note that this calculation de-facto represents an inverse problem to that addressed in this article.

2. Approximation Method

Following the rules of linear algebra it is much easier to get an insight into dynamical flow nature in the case of the piecewise-linear (PWL) systems. This kind of vector field is also very easy to be implemented as an electronic circuit, analog chaotic oscillator. From this point of view it can be realized using diodes, positive and negative resistors and independent dc current or voltage sources. Using Orcad Pspice terminology the negative imitance converters can be realized by using some combination of the ideal controlled sources E, G, H and F. The question is if hand-made PWL approximation can deformate the vector field so that the desired state space attractor is destructed. It is well known that the nonlinearity is responsible for folding of the dynamical flow while some eigenvalue with positive real part results into stretching, i.e. exponential divergence of the neighborhood trajectories. Both criterions must be preserved to get the bounded solution extremely sensitive to changes of the initial conditions. The basic presumption for the properties of the vector field is in that the number of fixed points and its stability index in the original and approximated system is exactly the same. These are necessary but not sufficient conditions that both PWL and smooth dynamical systems generate strange attractors with the same or close enough fractal dimension. Note that KYD is able to distinguish not only chaotic solution but also hyperchaos, quasiperiodic orbits, periodic or trivial fixed point solution.

Assume PWL approximation function expressed in the following compact form

$$h_n(y) = \alpha_0 + \sum_{i=1}^{n} \alpha_i |y - \beta_i| \tag{4}$$

where n is a total number of breakpoints. The quadratic nonlinearity in the state equations will be replaced by a less general two-segment PWL function

$$h_1(y) = \alpha_1 |y - \beta_1| \tag{5}$$

or four-segment PWL function

$$h_3(y) = \alpha_1 |y - \beta_1| + \alpha_2 |y - \beta_2| + \alpha_3 |y - \beta_3| \tag{6}$$

where coefficients α_i and β_i are a subject of optimization. It seems that more complicated PWL approximations need not to be considered. In other words, experiments show that hyperspace of six unknown parameters is large enough to successfully finish the searching procedure. Approximation (6) of the quadratic function y^2 is basically odd-symmetrical function with parameters

$$\alpha_1 = \alpha_3 = \frac{m_{out} - m_{in}}{2}, \qquad \alpha_2 = m_{in} \tag{7}$$

where m_{out} and m_{in} are slopes of PWL function in outer and inner segments respectively.

2.1 Properties of Fitness Function

Because of the problems mentioned above it would be probably better to transform searching for optimal PWL approximation into optimization task. The proposed fitness function will be the absolute value of difference between KYD of original dynamical system (reference value) and KYD of approximated system with actual PWL function. Thus in the case of optimal approximation both KYD are the same which means exactly the same spectrum of the LE. There are several obstacles which must be removed. Firstly the diverging solution must be penalized since this is not the motion we are looking for. This can be done by a basic test if the fiducial point is inside a large enough but finite volume element, for example sphere. Second problem is analysis of the system with slowly diverging solution. This is in accordance with the fact that regions of chaotic motion (in the sense of nonlinear function parameters) are usually narrow and surrounded by unstable solution. Due to the finite time interval of the calculation the fiducial point can be still inside a boundary sphere while equation (2) returns two positive numbers. Second trouble lies in what we call basin of attraction. There is no guarantee that observed strange attractor is a global attracting set, even in the case of nonlinear function symmetry. Although such a problem has been recognized and expected, it did not show in the process of verifying the proposed procedure by means of huge number of the polynomial dynamical systems.

2.2 Optimization Routine

Note that there is no general analytical formula for chaos quantification and gradient methods of optimization cannot be used. Thus some of the stochastic methods must be employed, for example genetic algorithm (GA), particle swarms (PS) or its combination is a good choice for this particular purpose. The serious drawback of these methods is in the necessity of large amount of repetitive objective function evaluation. This immediately leads to redundant calculations, very slow convergence ratio and the enormous time demands. To remove these disadvantages is the topic of current research. Since these routines are ideologically correct the core algorithm should not be improved. Recent progress in personal computer performance, especially invention of multi-core engines [17], allows the significant minimization of these time requirements. To do this it is necessary to rewrite the whole search procedure with respect to parallel processing and distributed calculations.

The proper choice of PWL function is perhaps even more important than fitness function evaluation itself. It will be continuous scalar function where the positions of the breakpoints and slopes in each state space region are variable parameters we are looking for. Using either GA or PS some restrictions and boundaries must be set. For example, if some slope of PWL function changes sign it can be hardly considered as a good approximation. Moreover it changes signs of the eigenvalues simultaneously and the chaotic nature of the desired dynamical behavior is lost. Also the virtual stability index of the invariant manifold in each segment of the vector field can be checked and the parameter set leading to its change should be omitted. Each attempt to find optimal PWL approximation deals with the fixed number of the breakpoints, i.e. only its location is changed during the run of search procedure.

3. Tests and Experimental Results

First of all, assume a class of the third-order autonomous deterministic dynamical systems with single equilibrium located at the origin [15]. The examples of such system can be written as case A

$$\dot{x} = -0.25y, \qquad \dot{y} = x + z, \qquad \dot{z} = x + y^2 - z, \quad (8)$$

where dot represents a derivative with respect to time. Straightforward linear analysis says that this system has a single equilibrium located at the origin. The characteristic polynomial yields $\lambda^3+\lambda^2+0.25\lambda+0.5=0$ and corresponding eigenvalues are $\lambda_{1,2}=0.067\pm0.59j$ and $\lambda_3=-1.134$. Reference values of KYD for each dynamical system with quadratic polynomial vector field approximated can be taken from the interesting review paper [18]. For case A this value is quite low 2.012 and leads to the optimal set of parameters in the sense of (6) and (7) $-\beta_1 = \beta_3 = 0.5$, $\beta_2 = 0$, $m_{in} = 0.53$ and $m_{out} = 1.74$. Similar dynamical system capable to produce chaotic motion is denoted as case B

$$\dot{x} = 2z, \qquad \dot{y} = -2y + z, \qquad \dot{z} = -x + y + y^2. \quad (9)$$

By performing fundamental calculation we can learn that there is only one fixed point at origin with the associated characteristic polynomial $\lambda^3+2\lambda^2+\lambda+4=0$ and roots become $\lambda_{1,2}=0.157\pm1.305j$ and $\lambda_3=-2.315$. Using GA with 2.037 as the desired value of KYD the proper PWL approximation has breakpoints $-\beta_1 = \beta_3 = 0.8$, $\beta_2 = 0$ and slopes $m_{in} = 1.4$, $m_{out} = 8.2$. The analogical computational approach will be done in the case C dynamical system, i.e.

$$\dot{x} = 0.5x - y, \qquad \dot{y} = x + z, \qquad \dot{z} = -z + y^2. \quad (10)$$

Assuming zero time derivatives there are two fixed points which are invariant under the flow located at $\mathbf{x}_1=(0, 0, 0)^T$ and $\mathbf{x}_2=(-4, -2, 4)^T$. Thus the characteristic polynomials are $\lambda^3+1.5\lambda^2+0.5\lambda+1=0$ and $\lambda^3+1.5\lambda^2-3.5\lambda+3=0$. Dynamical behavior in the neighborhood of these point is uniquely determined by two sets of the eigenvalues $\lambda_{1,2}=0.25\pm0.97j$, $\lambda_3=-1$ and $\lambda_{1,2}=0.97\pm0.538j$, $\lambda_3=-2.439$.

It seems that for reference KYD value 2.19 the proper approximation is (6) and (7) with parameters $-\beta_1=\beta_3=0.8$, $\beta_2=0$ and slopes $m_{in}=0.4$, $m_{out}=2.2$. The fourth system to be considered is denoted as case D

$$\dot{x} = -z + y^2, \qquad \dot{y} = x - y, \qquad \dot{z} = x + 0.84y. \quad (11)$$

Note that there is only one fixed point located at origin. The associated characteristic polynomial is $\lambda^3 + \lambda^2 + \lambda + 1.84 = 0$ and roots $\lambda_{1,2} = 0.155 \pm 1.175j$ and $\lambda_3 = -1.309$. For this system and KYD 2.19 one possible set of PWL function parameters is $-\beta_1 = \beta_3 = 0.75$, $\beta_2 = 0$ and slopes of the linear segments $m_{in} = 0.9$, $m_{out} = 5.31$. Another member of the systems under inspection is case E

$$\dot{x} = 2.7y + z, \qquad \dot{y} = -x + y^2, \qquad \dot{z} = x + y. \quad (12)$$

There are just two equilibria, the first one at $\mathbf{x}_1=(0, 0, 0)^T$ and the second one located at $\mathbf{x}_2=(1, -1, 2.7)^T$. Characteristic polynomials are $\lambda^3 + 1.7\lambda + 1 = 0$ and $\lambda^3 + 4\lambda^2 + 1.7\lambda - 3 = 0$. Three eigenvalues associated with the first and second fixed points of this system are $\lambda_{1,2} = 0.255 \pm 1.377j$, $\lambda_3 = -0.51$ together with $\lambda_{1,2}=-0.191\pm1.092j$, $\lambda_3=0.383$. If the value of KYD used in the fitness function equals 2.181 therefore PWL function parameters can be $-\beta_1=\beta_3=0.33$, $\beta_2=0$ and slopes $m_{in}=0.52$, $m_{out}=2.38$. The last example of the group of dynamical systems with single quadratic nonlinearity is the case F described by the following mathematical model

$$\dot{x} = -z, \qquad \dot{y} = x - y, \qquad \dot{z} = 3.1x + 0.48z + y^2. \quad (13)$$

There are just two fixed points located at $\mathbf{x}_1=(0, 0, 0)^T$ and $\mathbf{x}_2=(-3.1, -3.1, 0)^T$. Accordingly to this the characteristic polynomials can be obtained as $\lambda^3+1.52\lambda^2+2.62\lambda+3.1=0$ and $\lambda^3+1.52\lambda^2+2.62\lambda-3.1=0$. The roots can be established as $\lambda_{1,2}=-0.101\pm1.531j$, $\lambda_3=-1.317$ and $\lambda_{1,2}=-0.124\pm1.73j$, $\lambda_3=0.728$. Final PWL approximation routine deals with KYD equal to 2.179 and founded parameters $-\beta_1=\beta_3=1$, $\beta_2=0$ and slopes $m_{in}=0.04$, $m_{out}=4.87$. Note that systems A to F are different since there is no non-singular transformation of the coordinates which transforms one system to other. It should be also noted that it is sufficient to fix number of breakpoints on three and express each parameter we are looking for as a value less than eight and down to two decimal degrees (gene represented by three plus three bits). This has immediate consequences on the designed GA routine, namely binary encoding and consequently on chromosome length. In the case of studied dynamical systems the mutation percentage was set up on 5% and the best results were obtained by using tournament selection. Speaking in terms of PSO the best choice involves absorbing walls and dying bees which do not move. The weight functions and vector forces are as usual.

Before starting practical part of the work several types of the numerical analysis should be performed. One of the most important is visualization of the attraction region for typical chaotic attractor; see Fig. 1, Fig. 2 and Fig. 3. In these pictures, Mathcad and build-in fourth-order Runge-Kutta method with integration time step 0.01, length 1000 and initial condition step size $\Delta_x=\Delta_z=0.1$ has been utilized and centered around origin. Due to the huge time demands requested by full three-dimensional view and its difficult transparent visualization of these subspaces only plane fragments are provided. The difference between individual dynamical flows near the y=0 cross section is demonstrated via Fig. 4. The comparison between global chaotic motion of the original systems and approximated mathematical models is provided by means of Fig. 5 and 6.

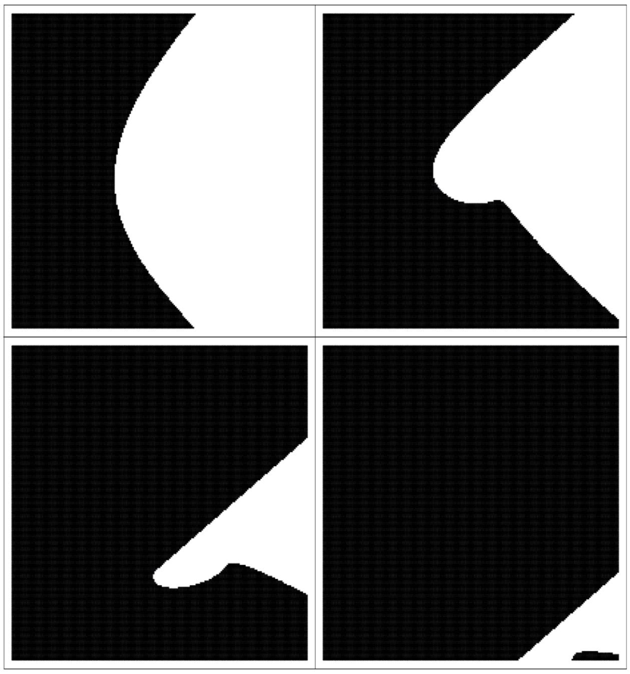

Fig. 1. Plane fragments of the basin of attraction for case B approximated dynamical system, namely *y*=-3, *y*=0, *y*=3 and *y*=6.

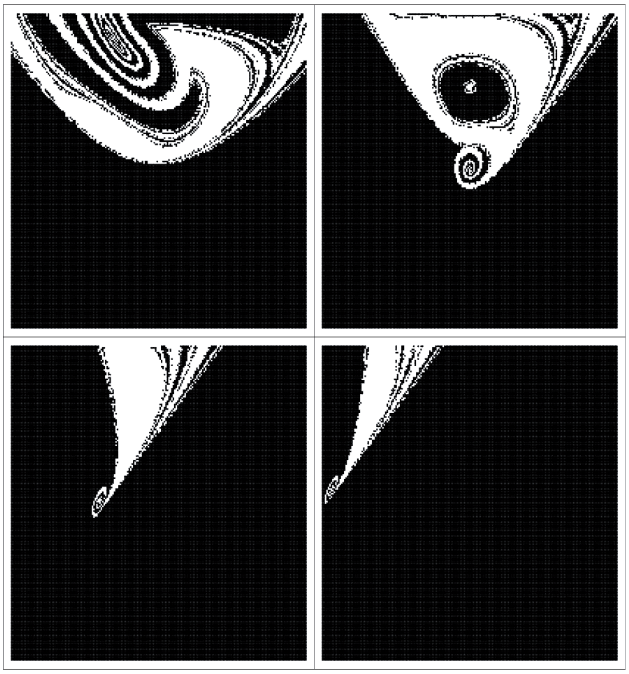

Fig. 2. Plane fragments of the basin of attraction for case D approximated dynamical system, namely y=-3, y=0, y=3 and y=6.

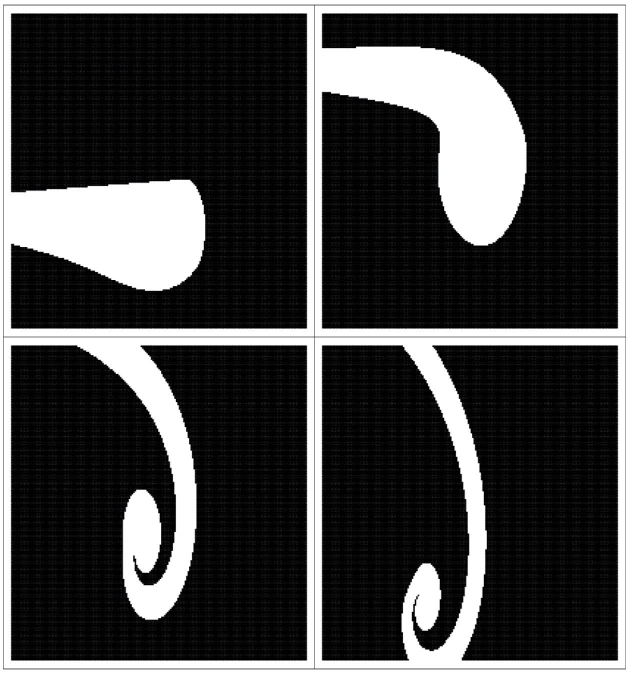

Fig. 3. Plane fragments of the basin of attraction for case F approximated dynamical system, namely y=-3, y=0, y=3 and y=6.

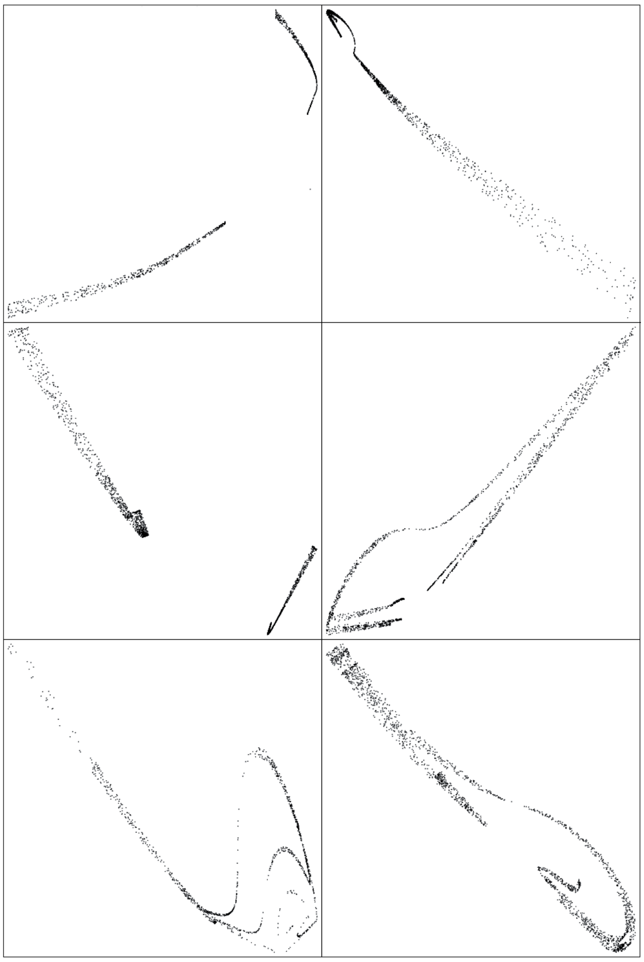

Fig. 4. Poincaré sections for case A to F approximated dynamical system, trajectory consists of 10^5 points with step size reduced down to 0.001.

4. Circuitry Implementation

Each dynamical system presented in this paper has some circuitry implementation which is canonical in the sense of minimal number of passive and active elements. Such network structures cannot be obtained by a circuit synthesis based on the integrator block schematic [19]. This approach is straightforward but requires many active circuit elements. Realized dynamical system should be more likely decomposed into nonlinear and linear part and classical circuit synthesis approach should be employed for the latter case. The basic active elements for the dynamical system realization are voltage follower and inverter. Nowadays it is possible to significantly minimize necessary active and passive elements by utilizing modern functional block which internal chip structure performs several mathematical operations simultaneously. Moreover we are not restricted only to the commercially available devices but there are hypothetical as well [20]. Among all these active elements we prefer devices with voltage input and current output because the state variables will be voltages across grounded capacitors due to the easy measurability. It seems that the most promising example belonging to such class is three-terminal general current conveyor (GCC) [21] described by the following hybrid equations

$$V_X = \alpha \cdot V_Y, \qquad I_Y = \beta \cdot I_X, \qquad I_Z = \gamma \cdot I_X, \quad (14)$$

especially its off-the-shelf member called positive (AD844) and negative (EL2082) second generation current conveyor with $\alpha = 1$, $\beta = 0$, $\gamma = \pm 1$. Another useful building block for circuit synthesis is multiple-output transconductance amplifiers (MOTA) [22] with differential voltage input and multiple current outputs

$$I_{Z1} = g_m \cdot (V_X - V_Y), \qquad I_{Z2} = \pm g_m \cdot (V_X - V_Y). \quad (15)$$

Although there is a lack of commercially available MOTA devices it can be constructed by using a parallel connection of appropriate number of single-output transconductors, for example diamond transistors [23]. Last but not least, a rich variety of the signal processing applications can be done by utilizing differential-voltage current conveyor (DVCC) [24] represented by equations

$$I_{Y1} = 0, \quad I_{Y2} = 0, \quad V_X = V_{Y1} - V_{Y2}, \quad I_Z = I_X. \quad (16)$$

Note that this DVCC basically has only one current output. There are multiple-output modifications in the sense that I_X is copied to multiple output nodes. The idealized circuit model for each active device mentioned above can be directly derived using describing equations. The non-ideal properties necessary for completing level 3 or 4 models can be found in the literature.

The linear part of the vector field can be implemented by using floating or grounded linear bilateral two-terminals like capacitors, inductors and resistors. These elements exist in both positive and negative modifications but negativity suggests at least one additional active element working in its linear regime of operation. It is worth nothing that the problem of circuit synthesis has non-unique solutions, i.e. we can found different circuits which are described by the same set of the differential equations. In any way we get the normalized values of resistors, capacitors and inductors which can be hardly used in practice. Thus the time and impedance rescaling must be done. The choice of rescaling factor, namely the time constant, is not arbitrary but upper limited by a roll-off transfer nature of the used active devices. For reasonable value of both scaling factors the parasitic properties of the used active devices have negligible side effects which occur as the error terms in the describing differential equations and can be omitted.

In the case of original mathematical models with polynomial nonlinearity the linear part of the vector field can be realized by the same subcircuit. The first and correct idea how to deal with the quadratic polynomial is to use four-quadrant analog multiplier such as AD633 (voltage mode) or EL4083 (current mode). The essential property of these integrated circuits which should be checked before using is the range of linear operation region, i.e. output voltage or current saturation.

For the nonlinear part of the vector field we use two-ports with desired PWL transfer characteristics. Such a curve can be easily realized by a cascade of the ideal diodes represented as switches controlled by the input terminal voltage. This conception allows independent adjusting the breakpoints (V_1, V_2, etc.) and slopes (R_1, R_2, etc.) of the PWL function, see Fig. 7. For additional segments of PWL function it is sufficient to add diode, dc voltage source and resistor composite. Note that the breakpoints and slopes in circuit simulator schematic editor can be considered as global parameters. It is evident that using this conception we have condition $m_{in} < m_{out}$ which is always satisfied. This holds in general for arbitrary number of segments, it means lower slope for segment closer to zero. Attempt to obtain the canonical network for case A, C, F dynamical system is shown in Fig. 8, Fig. 9 and Fig. 10 respectively together with real values of circuit elements. The Orcad Pspice time domain analysis of three examples of such circuits transformed into plane projections is provided by means of Fig. 11, Fig. 12 and Fig. 13.

Sometimes it is useful to plot time-domain difference between original and approximated PWL dynamics or error system. Unfortunately a motion of chaotic system is a subject of dramatic changes also in the case of tiny changes of the internal system parameters. That is why the state attractor as a difference between two strange attractors always fills the fractal state space volume in quite a short time; no matter how precise the approximation is. This also occurs if two identical mathematical models with slightly different parameters are analyzed in time domain and compared. That is the reason why chaotic dynamics is difficult or almost impossible to utilize for modulation. PWL approximation is considered as appropriate if desired state space attractor properties are preserved.

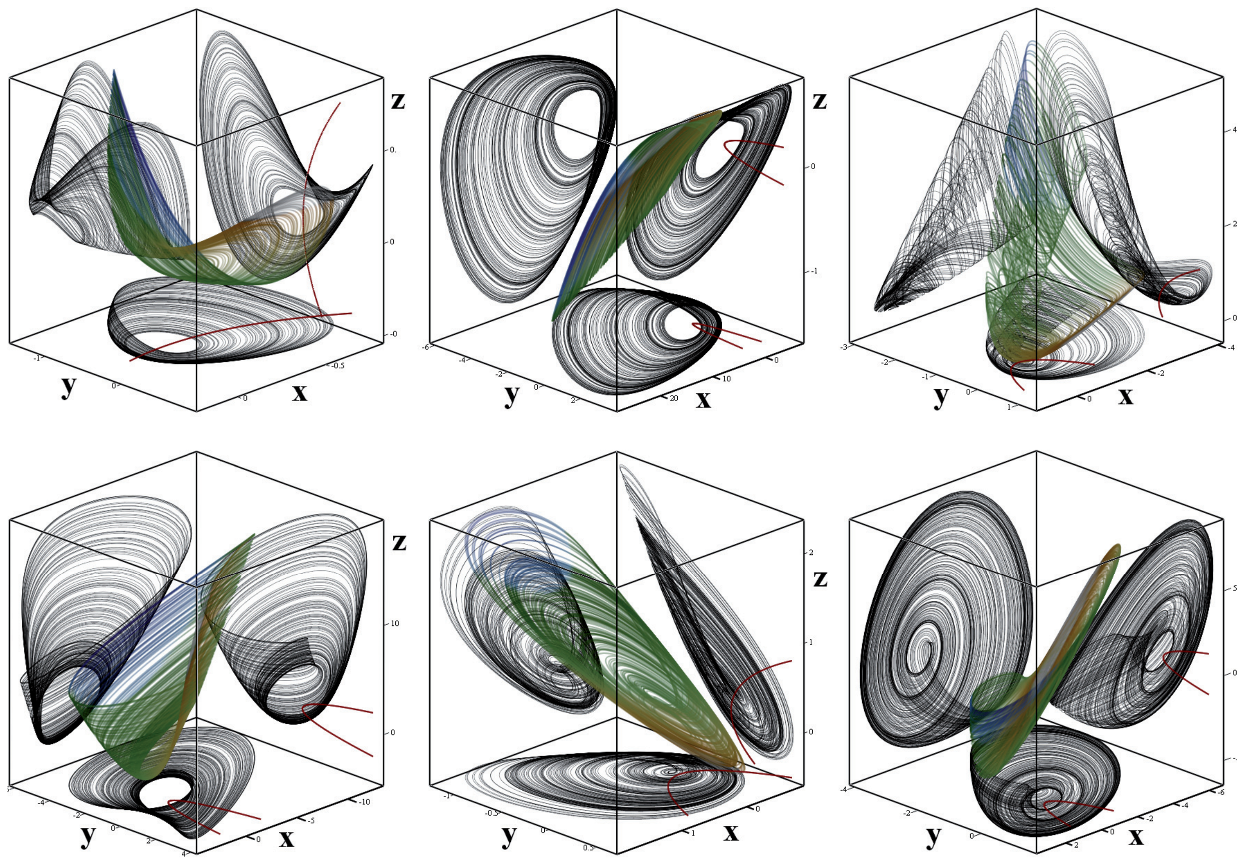

Fig. 5. 3D perspective views on the typical strange attractors of the original dynamical systems.

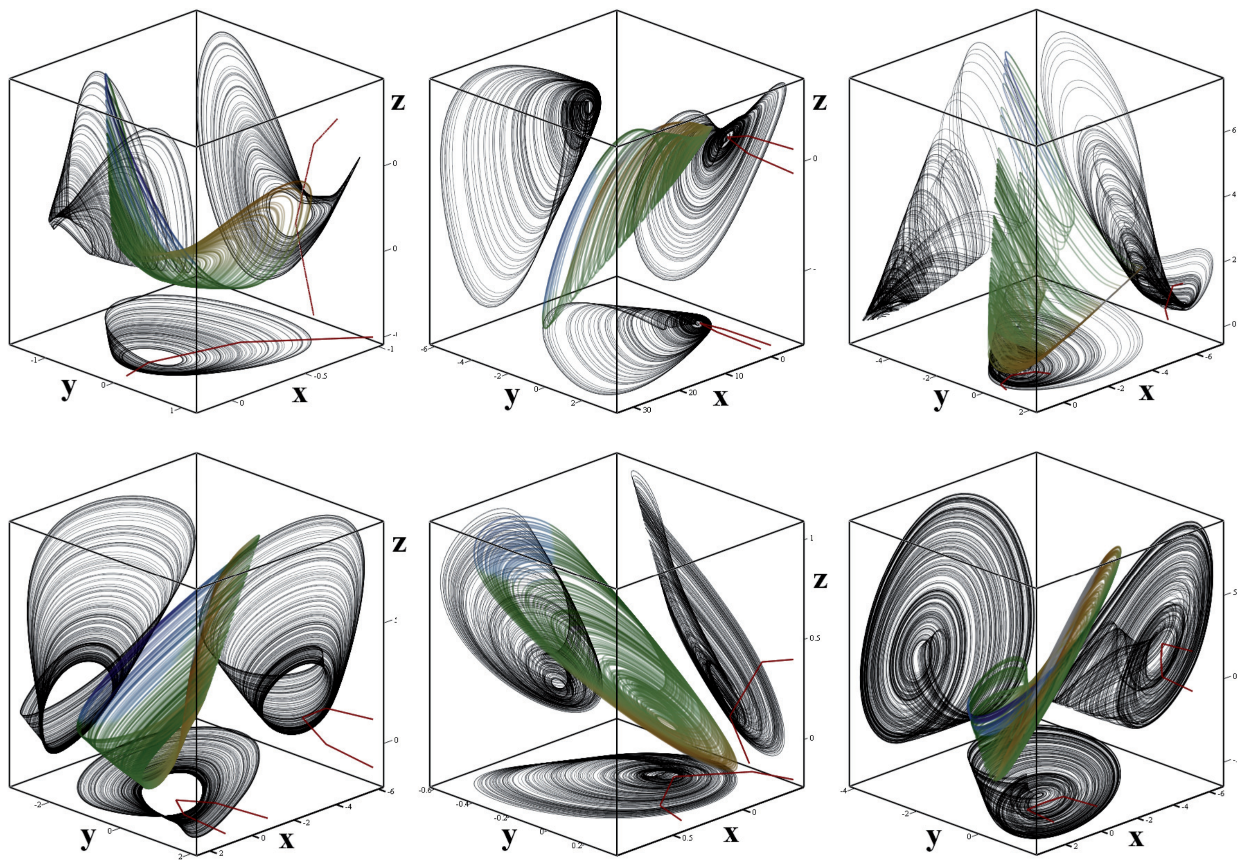

Fig. 6. 3D perspective views on the equivalent strange attractors of the approximated dynamical systems.

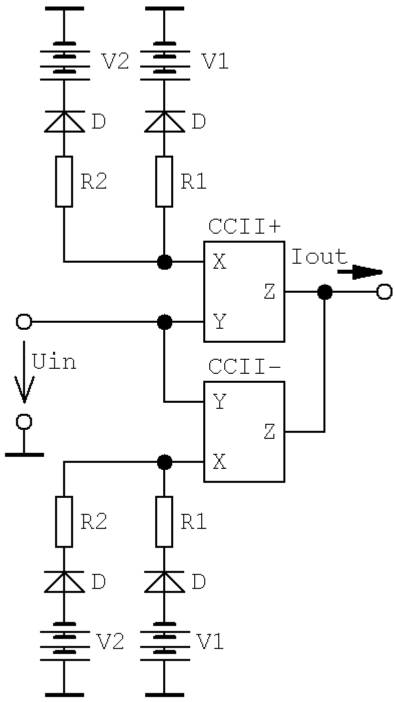

Fig. 7. Circuitry realization of odd-symmetrical piecewise-linear two-port with voltage input and current output.

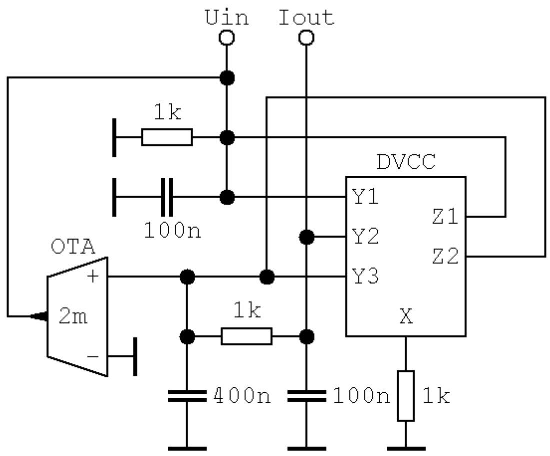

Fig. 8. Circuitry implementation of case A dynamical system.

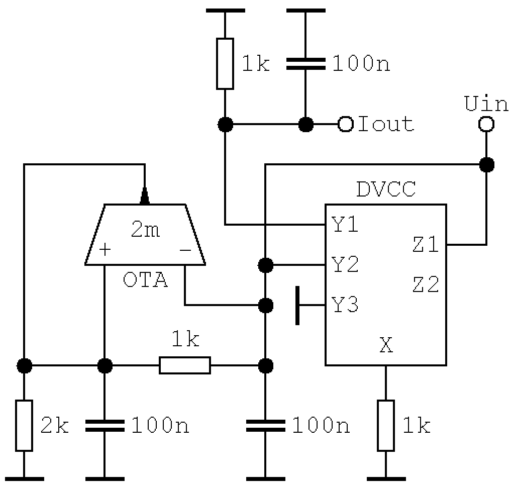

Fig. 9. Circuitry implementation of case C dynamical system.

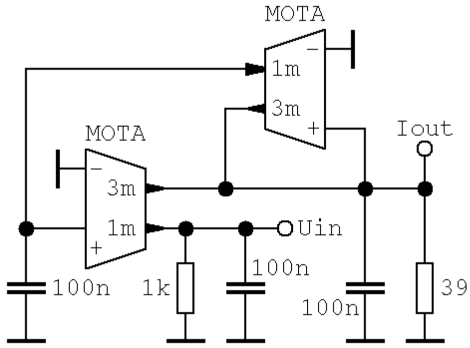

Fig. 10. Circuitry implementation of case F dynamical system.

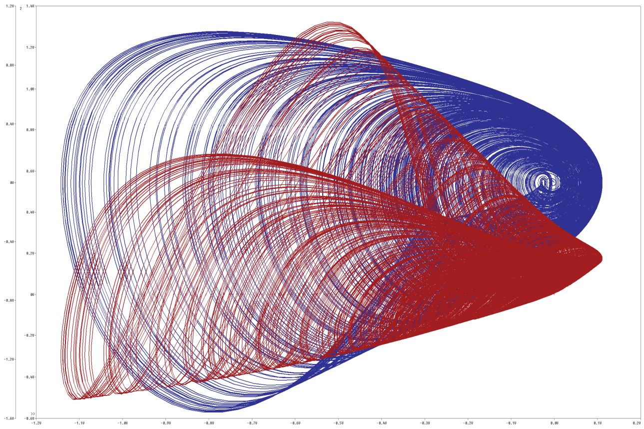

Fig. 11. Orcad Pspice simulation of case A system, plane projections.

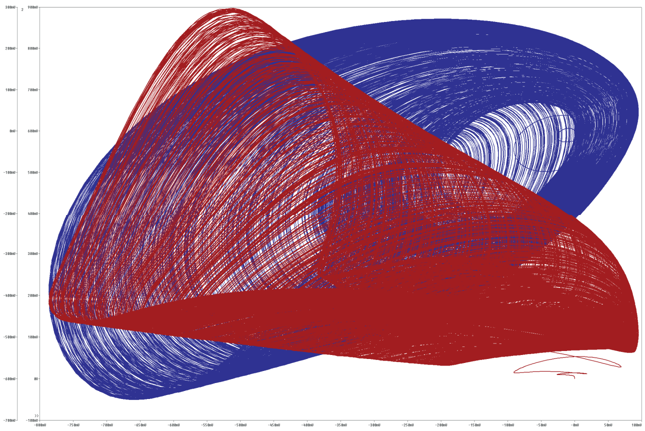

Fig. 12. Orcad Pspice simulation of case C system, plane projections.

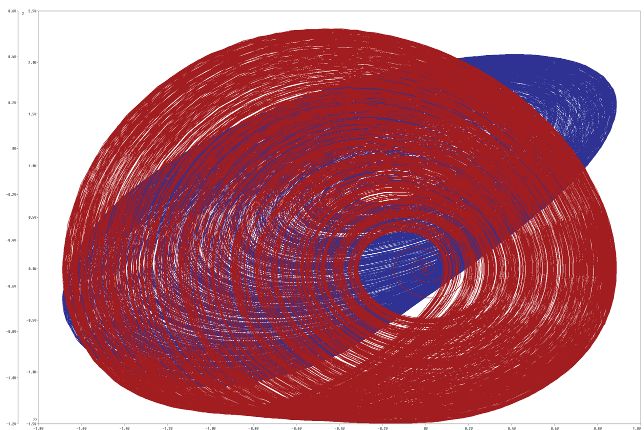

Fig. 13. Orcad Pspice simulation of case F system, plane projections.

5. Conclusion

The method of utilizing stochastic optimization methods for PWL approximation proposed in this paper is universal and can be used for the higher-order dynamical systems with arbitrary number of the linear segments. First concepts have been already published in the preliminary paper [25]. These problems are closely related to the solution of the lumped circuits as long as we are able to describe them by a system of the differential equations. PWL approximation is essential for simplification of synthesized circuit since the analog multiplier can be replaced by the diode limiters. One can create the chaotic oscillators ready for high frequency applications, i.e. devices recently highly sought for various communication techniques. The continued verification of the proposed procedure by means of the cubic polynomial nonlinearities in the describing state equations as well as searching for the better chaos quantifier are topics for further research. Some interesting and practical examples of the simple chaotic systems can be found in monograph [26]. The optimization approach itself to solve the problem of correct smooth vector field approximation is mandatory since there is no relation between least mean square error (LMS) estimation and the structural geometrical stability of the desired chaotic attractor. In fact uniform distribution of LMS over the scale of the chaotic attractor provides either a very rough approximation or redundant number of the breakpoints leading to the complexity of the network.

The chaos is multidisciplinary science which can be observed also in ecology, economics, medicine and biology [27]. Thus there are many reasons why solve challenges and topics from this interesting areas.

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