The Study of Properties of n-D Analytic Signals and Their Spectra in Complex and Hypercomplex Domains

Kajetana Marta SNOPEK

Institute of Radioelectronics, Warsaw University of Technology, Nowowiejska 15/19, 00-665 Warsaw, Poland

snopek@ire.pw.edu.pl

Abstract. *In the paper, two various representations of a n-dimensional (n-D) real signal $u(x_1,x_2,\dots,x_n)$ are investigated. The first one is the n-D complex analytic signal $\psi(x_1,x_2,\dots,x_n)$ with a single-orthant spectrum defined by Hahn in 1992 as the extension of the 1-D Gabor's analytic signal. It is compared with two hypercomplex approaches: the known n-D Clifford analytic signal ψ_{Cl} and the Cayley-Dickson analytic signal ψ_{CD} defined by the Author in 2009. The signal-domain and frequency-domain definitions of these signals are presented and compared in 2-D and 3-D. Some new relations between the spectra in 2-D and 3-D hypercomplex domains are presented. The paper is illustrated with the example of a 2-D separable Cauchy pulse.*

Keywords

Analytic signals, Cayley-Dickson algebra, Clifford algebra, Clifford Fourier transform, Quaternion Fourier transform, Octonion Fourier transform.

1. Introduction

The theory of complex (CS) and hypercomplex (HS) signals and their Fourier transforms (FTs) is a subject of many publications involving mathematicians and engineers working in different fields, e.g., in digital signal processing, especially in color image processing [1]-[8]. Different approaches are applied and the choice of a method is arbitrary.

In color image processing, reduced quaternions and reduced commutative biquaternions [8] are used as representations of a RGB image. Its spectrum is calculated using the Quaternion FT introduced and used by Ell [3] or its discrete version called Quaternion Discrete FT (QDFT) [1] which opens up a very wide range of possible applications [4], [5], [9]. In [4] for example, the problem of estimation of the motion characteristics within a time-varying color image scene is investigated. It has been shown that using hypercomplex (quaternion) approach results in lower computational cost and better performance in presence of different distortions. Another problem is the edge detection in color images using filters based on convolution with quaternion masks or the Discrete Quaternion Correlation function [5], [9]. The hypercomplex representation of a color image allows easy detecting objects with the same shape, size, color and brightness as the reference pattern (or presenting only two common features, like shape and color or shape and brightness).

In this paper, we put attention on n-D analytic signals and show some relations in complex and hypercomplex domains for $n = 2$ and $n = 3$. Let us recall that in 2-D, an analytic signal is a complex/hypercomplex representation of a 2-D real signal (an image) $u(x_1, x_2)$. In 3-D applications, a real signal $u(x_1, x_2, x_3)$, i.e. a volume image, can also be replaced with the corresponding complex/hypercomplex analytic signal.

In Section 2, we introduce notions of n-D analytic signals using complex and hypercomplex delta distributions defined in Secion 1. The Section 4 includes frequency-domain definitions of complex/hypercomplex analytic signals exploiting different Fourier transformations recalled in Section 3. The main contributions of the paper are: the formula relating the Octonion Fourier Transform with the 3-D complex Fourier Transform (derived and proved) and other relations concerning the hypercomplex spectra in 2-D and 3-D.

2. Complex and Hypercomplex n-D Delta Distributions

2.1 The n-D Complex Delta Distribution

The n-D complex delta distribution (CDD) has been defined in 1996 [10] as the extension of the 1-D CDD: $\Psi^{\delta}(t) = \delta(t) + j / \pi t$, widely used in the theory of signals and systems and also in quantum physics [11], [12]. Given a n-D variable $\boldsymbol{x} = (x_1,\dots,x_n)$, we have

$$\Psi^{\delta}(\boldsymbol{x}) = \prod_{i=1}^{n}\left[\delta(x_i) + \frac{e_1}{\pi x_i}\right] = \prod_{i=1}^{n}\left(\delta_i + \frac{e_1}{\pi x_i}\right). \tag{1}$$

For $n = 2$ and 3, we obtain respectively

$$\Psi^{\delta}\left(x_1,x_2\right)=\delta_1\delta_2-\frac{1}{\pi^2 x_1 x_2}+e_1\left[\frac{\delta_1}{\pi x_2}+\frac{\delta_2}{\pi x_1}\right], \quad (2)$$

$$\begin{aligned}\Psi^{\delta}&\left(x_1,x_2,x_3\right)\\ &=\delta_1\delta_2\delta_3-\frac{\delta_1}{\pi^2 x_2 x_3}-\frac{\delta_2}{\pi^2 x_1 x_3}-\frac{\delta_3}{\pi^2 x_1 x_2}\\ &+e_1\left[\frac{\delta_2\delta_3}{\pi x_1}+\frac{\delta_1\delta_3}{\pi x_2}+\frac{\delta_1\delta_2}{\pi x_3}-\frac{1}{\pi^3 x_1 x_2 x_3}\right].\end{aligned} \quad (3)$$

2.2 The n-D Hypercomplex Delta Distribution

The general definition of the n-D hypercomplex delta distribution (HDD) can be found in [13]. Its form depends on a chosen algebra of hypercomplex numbers. For example in the Clifford algebra [14], it is given by

$$\Psi^{\delta}_{\mathrm{Cl}}\left(\boldsymbol{x}\right)=\prod_{i=1}^{n}\left[\delta\left(x_i\right)+e_i/\pi x_i\right]=\prod_{i=1}^{n}\left(\delta_i+e_i/\pi x_i\right) \quad (4)$$

Where e_i are elements of the basis of the Clifford algebra: $\left\{e_{i_1}e_{i_2}\ldots e_{i_k}: 1\le i_1<i_2<\ldots<i_k\le n, 0\le k\le n\right\}$. In the Cayley-Dickson algebra, we have proposed in [13] the following definition of the n-D HDD:

$$\Psi^{\delta}_{\mathrm{CD}}\left(\boldsymbol{x}\right)=\prod_{i=0}^{n-1}\left[\delta\left(x_i\right)+e_{2^i}/\pi x_i\right]=\prod_{i=0}^{n-1}\left(\delta_i+e_{2^i}/\pi x_i\right). \quad (5)$$

Let us write (4) and (5) for n = 2:

$$\Psi^{\delta}_{\mathrm{Cl}}\left(x_1,x_2\right)=\delta_1\delta_2+e_1\frac{\delta_2}{\pi x_1}+e_2\frac{\delta_1}{\pi x_2}+\left(e_1e_2\right)\frac{1}{\pi^2 x_1 x_2}, \quad (6)$$

$$\Psi^{\delta}_{\mathrm{CD}}\left(x_1,x_2\right)=\delta_1\delta_2+e_1\frac{\delta_2}{\pi x_1}+e_2\frac{\delta_1}{\pi x_2}+e_3\frac{1}{\pi^2 x_1 x_2} \quad (7)$$

and n = 3 respectively:

$$\begin{aligned}\Psi^{\delta}_{\mathrm{Cl}}&\left(x_1,x_2,x_3\right)\\ &=\delta_1\delta_2\delta_3+e_1\frac{\delta_2\delta_3}{\pi x_1}+e_2\frac{\delta_1\delta_3}{\pi x_2}+\left(e_1e_2\right)\frac{\delta_3}{\pi^2 x_1 x_2}+e_3\frac{\delta_1\delta_2}{\pi x_3}\\ &+\left(e_1e_3\right)\frac{\delta_2}{\pi^2 x_1 x_3}+\left(e_2e_3\right)\frac{\delta_1}{\pi^2 x_2 x_3}+\left(e_1e_2e_3\right)\frac{1}{\pi^3 x_1 x_2 x_3},\end{aligned} \quad (8)$$

$$\begin{aligned}\Psi^{\delta}_{\mathrm{CD}}&\left(x_1,x_2,x_3\right)\\ &=\delta_1\delta_2\delta_3+e_1\frac{\delta_2\delta_3}{\pi x_1}+e_2\frac{\delta_1\delta_3}{\pi x_2}+e_3\frac{\delta_3}{\pi^2 x_1 x_2}+e_4\frac{\delta_1\delta_2}{\pi x_3}\\ &+e_5\frac{\delta_2}{\pi^2 x_1 x_3}+e_6\frac{\delta_1}{\pi^2 x_2 x_3}+e_7\frac{1}{\pi^3 x_1 x_2 x_3}.\end{aligned} \quad (9)$$

We see in (6) and (8) that the 2-D (resp. 3-D) delta distribution defined in the Clifford algebra has the form of a split-quaternion (resp. split-octonion) signal while applying in (7) and (9) the multiplication rules of the Cayley-Dickson algebra, we obtain a quaternion (resp. octonion) signal [14].

The general formulas given by (1), (4) and (5) will be used next to define n-D complex/hypercomplex analytic signals in the $\boldsymbol{x}$–domain (signal-domain). Different forms of hypercomplex delta distributions in (6), (7) and (8), (9) will affect the definitions of hypercomplex analytic signals. It will be shown that the Cayley-Dickson analytic signal and the Hahn's analytic signal are compatible in terms of their energy expressed by the standard Euclidean norm.

3. Signal-Domain Definitions of n-D Analytic Signals

As it has been mentioned above, the notion of the n-D CDD/HDD is used to define the n-D complex/hypercomplex analytic signal $\psi(\boldsymbol{x})$ in the n-D signal-domain. Given a real n-D signal $u(\boldsymbol{x})$ = $u(x_1, \ldots, x_n)$, the n-D analytic signal is defined by the n-D convolution:

$$\psi\left(\boldsymbol{x}\right)=u\left(\boldsymbol{x}\right)*\ldots*\Psi^{\delta}\left(\boldsymbol{x}\right). \quad (10)$$

Introducing (1), (4) and (5) into (10), we get three different definitions of n-D analytic signals. The first one is the n-D complex analytic signal with single-orthant spectrum defined in [15] as the extension of the Gabor's 1-D analytic signal (the orthant is a half-axis in 1-D, a quadrant in 2-D, an octant in 3-D, etc.). Secondly, we get analytic signals defined in the Clifford and Cayley-Dickson algebras and generally classified as hypercomplex. Note that the appropriate change of signs in definitions (1), (4), (5) yields analytic signals with spectra in other orthants. Here, we consider only signals with spectra in the 1st orthant, i.e., ($f_1>0$, $f_2>0$) in 2-D, ($f_1>0$, $f_2>0$, $f_3>0$) in 3-D. All presented approaches differ as it will be shown for n = 2 and n = 3. Fig. 1 shows the labeling of orthants in two and three dimensions.

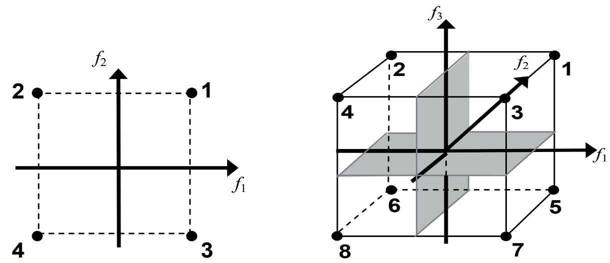


Fig. 1. Labelling of orthants in the 2-D and 3-D frequency space.

3.1 Analytic Signals in 2-D

Let us consider the case of the 2-D real signal $u(\boldsymbol{x})$, $\boldsymbol{x}$ = (x_1, x_2). Directly from (2) and (10), we get the definition of the 2-D complex signal (CS) with the spectrum in the 1st quadrant:

$$\begin{aligned}\psi\left(\boldsymbol{x}\right)&=u**\left[\delta_1\delta_2-\frac{1}{\pi^2 x_1 x_2}+e_1\left(\frac{\delta_1}{\pi x_2}+\frac{\delta_2}{\pi x_1}\right)\right]\\ &=u-v+e_1\left(v_1+v_2\right)\end{aligned} \quad (11)$$

where v is the total Hilbert transform of u, and v_1, v_2 are partial Hilbert transforms w.r.t. x_1 and x_2 [15], [16]. The same functions appear in definitions of the 2-D Clifford and Cayley-Dickson analytic signals, respectively $\psi_{\mathrm{Cl}}(\boldsymbol{x})$ and $\psi_{\mathrm{CD}}(\boldsymbol{x})$. Let us mention that the last one is known as the quaternion analytic signal [17]. We have:

$$\psi_{\mathrm{Cl}}(\boldsymbol{x}) = u + e_1 v_1 + e_2 v_2 + (e_1 e_2) v, \tag{12}$$

$$\psi_{\mathrm{CD}}(\boldsymbol{x}) = u + e_1 v_1 + e_2 v_2 + e_3 v. \tag{13}$$

Fig. 2 shows the example of the 2-D separable Cauchy signal:

$$u(x_1, x_2) = ab \Big/ \left(a^2 + (x_1 - c)^2\right)\left(b^2 + (x_2 - d)^2\right) \tag{14}$$

with parameters: $a = 1$, $b = 2$, $c = 0.5$, $d = 1$. Next Figs 3-5 present its total and partial Hilbert transforms given by

$$v(x_1, x_2) \\ = (x_1 - c)(x_2 - d) \Big/ \left(a^2 + (x_1 - c)^2\right)\left(b^2 + (x_2 - d)^2\right), \tag{15}$$

$$v_1(x_1, x_2) \\ = b(x_1 - c) \Big/ \left(a^2 + (x_1 - c)^2\right)\left(b^2 + (x_2 - d)^2\right), \tag{16}$$

$$v_2(x_1, x_2) \\ = a(x_2 - d) \Big/ \left(a^2 + (x_1 - c)^2\right)\left(b^2 + (x_2 - d)^2\right). \tag{17}$$

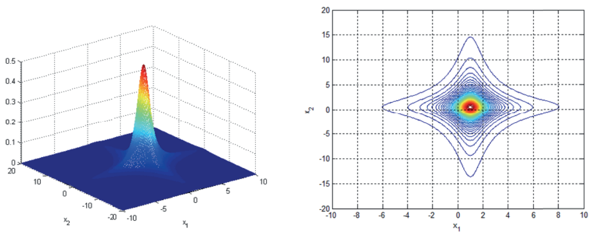


Fig. 2. Mesh (left) and contour (right) plots of the 2-D Cauchy signal $u(x_1, x_2)$: $a = 1$, $b = 2$, $c = 1$, $d = 0.5$.

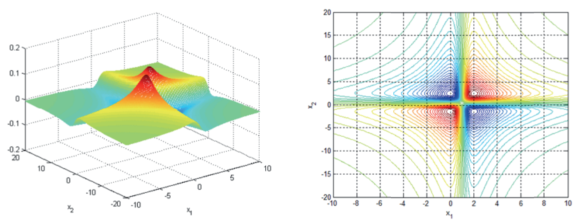


Fig. 3. Mesh (left) and contour (right) plots of $v(x_1, x_2)$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 0.5$, $d = 1$.

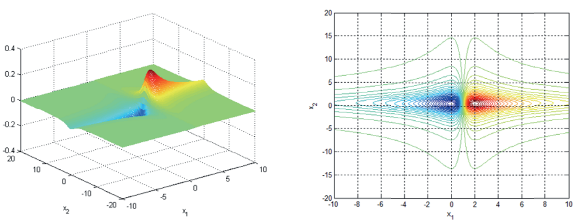


Fig. 4. Mesh (left) and contour (right) plots of $v_1(x_1, x_2)$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 0.5$, $d = 1$.

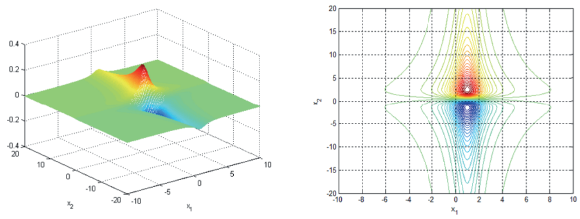


Fig. 5. Mesh (left) and contour (right) plots of $v_2(x_1, x_2)$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 0.5$, $d = 1$.

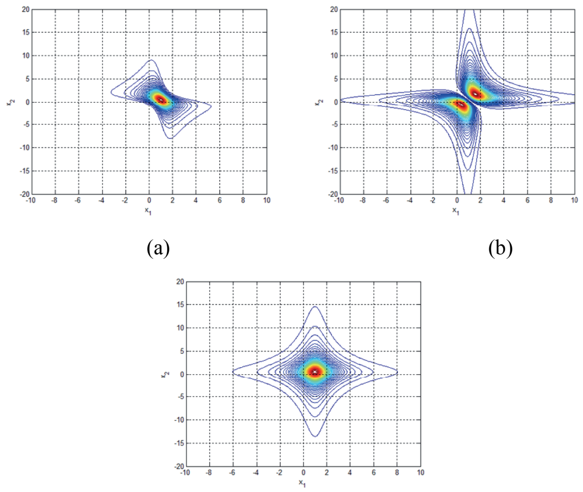
It is observed that in general squared norms (squared local amplitudes) of signals (11)-(13) differ since we have

$$\|\psi\|^2 = (u - v)^2 + (v_1 + v_2)^2, \tag{18}$$

$$\|\psi_{\mathrm{Cl}}\|^2 = u^2 + v^2 + v_1^2 - v_2^2, \tag{19}$$

$$\|\psi_{\mathrm{CD}}\|^2 = u^2 + v^2 + v_1^2 + v_2^2. \tag{20}$$

Note the minus sign appearing in (19) due to multiplication rules in the Clifford algebra [14]. The integration of (18)-(20) in the signal domain yields the energy of a given 2-D analytic CS/HS. However, for some signals, e.g. separable 2-D Cauchy or Gaussian signals [18]): $v_1 v_2 - uv = 0$, and in consequence, the energies of ψ and ψ_{CD} are equal. Figs 6 and 7 show the contour plots of 2-D squared local amplitudes of signals (11)-(13). It can be noticed that the local amplitude (18) of the Cayley-Dickson analytic signal is the same as (20) and differs from (19) (Fig. 7).



(a) (b)

(c)

Fig. 6. The local amplitude of analytic signal (11): (a) $(u - v)^2$, (b) $(v_1 + v_2)^2$, (c) $(u - v)^2 + (v_1 + v_2)^2$ (Remark: exactly the same picture represents the local amplitude of the Cayley-Dickson analytic signal (13)).

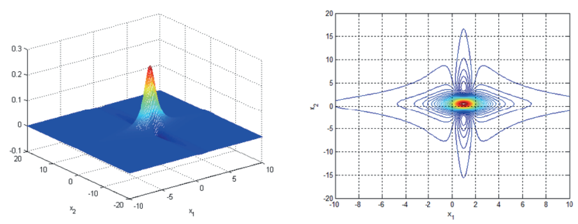


Fig. 7. The local amplitude of the 2-D Clifford analytic signal (12).

3.2 Analytic Signals in 3-D

Given a 3-D real signal $u(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, x_3)$, we present definitions of three 3-D analytic signals: the complex one, the Clifford signal and the octonion signal introduced in [19]. Using (3), (8)-(10), we arrive to the following definitions:

$$\psi(\boldsymbol{x}) = u - v_{23} - v_{13} - v_{12} + e_1(v_1 + v_2 + v_3 - v), \quad (21)$$

$$\begin{aligned}\psi_{\mathrm{Cl}}(\boldsymbol{x}) = u + e_1 v_1 + e_2 v_2 + (e_1 e_2) v_{12} + e_3 v_3 \\ + (e_1 e_3) v_{13} + (e_2 e_3) v_3 + (e_1 e_2 e_3) v,\end{aligned} \quad (22)$$

$$\begin{aligned}\psi_{\mathrm{CD}}(\boldsymbol{x}) = u + e_1 v_1 + e_2 v_2 + e_3 v_{12} + e_4 v_3 \\ + e_5 v_{13} + e_6 v_{23} + e_7 v.\end{aligned} \quad (23)$$

It can be noticed that (22) has the form of a split-octonion signal, while (23) has the octonion structure. In consequence, their energies in general differ since

$$\|\psi\|^2 = (u - v_{23} - v_{13} - v_{12})^2 + (v_1 + v_2 + v_3 - v)^2, \quad (24)$$

$$\|\psi_{\mathrm{Cl}}\|^2 = u^2 + v_1^2 + v_2^2 - v_{12}^2 + v_3^2 - v_{13}^2 - v_{23}^2 + v^2, \quad (25)$$

$$\|\psi_{\mathrm{CD}}\|^2 = u^2 + v_1^2 + v_2^2 + v_{12}^2 + v_3^2 + v_{13}^2 + v_{23}^2 + v^2. \quad (26)$$

For separable 3-D signals, similarly to the 2-D case, the expressions (24) and (26) coincide and as a result, the energies of analytic complex and octonion signals are equal.

4. Fourier Transforms in Complex and Hypercomplex Domains

The commonly used method of analysis of the frequency content of a n-D (real/complex/hypercomplex) signal $h(\boldsymbol{x})$ is the Fourier transform $H(\boldsymbol{f})$. Here, we describe two other approaches known as the Clifford Fourier transform (CFT) - $H_{\mathrm{Cl}}(\boldsymbol{f})$ [20], [21] and the hypercomplex Cayley-Dickson Fourier transform (CD-FT) - $H_{\mathrm{CD}}(\boldsymbol{f})$, introduced in [13] and named the Octonion Fourier transform (OFT) in 3-D [19]. We have

$$H(\boldsymbol{f}) = \int_{\mathrm{R}^n} h(\boldsymbol{x}) \prod_{i=1}^{n} \exp(-e_1 2\pi f_i x_i) \, d^n \boldsymbol{x}, \quad (27)$$

$$H_{\mathrm{Cl}}(\boldsymbol{f}) = \int_{\mathrm{R}^n} h(\boldsymbol{x}) \prod_{i=1}^{n} \exp(-e_i 2\pi f_i x_i) \, d^n \boldsymbol{x}, \quad (28)$$

$$H_{\mathrm{CD}}(\boldsymbol{f}) = \int_{\mathrm{R}^n} h(\boldsymbol{x}) \prod_{i=0}^{n-1} \exp\left(-e_{2^i} 2\pi f_i x_i\right) d^n \boldsymbol{x}. \quad (29)$$

Evidently, for 1-D signals, all three definitions are exactly the same. For $n = 2$, the CFT is known as the Quaternion Fourier transform (QFT), precisely the Right-Side (RS-QFT). Note that in literature, e.g. [9], two other forms of the QFT are applied, i.e. the Left-Side (LS-QFT), and the two-side (TS-QFT). Here, we consider only the RS-QFT and TS-QFT. For $n = 2$, the definitions (28) and (29) coincide. The difference appear for $n \geq 3$ as it will be shown in Section 5.

4.1 Quaternion (2-D) Fourier Transforms

As mentioned above, the QFT can be defined in two different forms: the RS-QFT - $H_{\mathrm{RS}}(\boldsymbol{f})$, and TS-QFT – $H_{\mathrm{TS}}(\boldsymbol{f})$. For real signals, both definitions are equivalent but for complex/hypercomplex signals the order of imaginary units under the integral cannot be changed (the algebra of quaternions is not commutative). We have

$$\text{RS – QFT:} \quad H_{\mathrm{RS}}(\boldsymbol{f}) = \int_{\mathrm{R}^2} h(\boldsymbol{x}) \mathrm{e}^{-e_1 2\pi f_1 x_1} \, \mathrm{e}^{-e_2 2\pi f_2 x_2} d^2 \boldsymbol{x}, \quad (30)$$

$$\text{TS – QFT:} \quad H_{\mathrm{TS}}(\boldsymbol{f}) = \int_{\mathrm{R}^2} \mathrm{e}^{-e_1 2\pi f_1 x_1} \, h(\boldsymbol{x}) \mathrm{e}^{-e_2 2\pi f_2 x_2} d^2 \boldsymbol{x}. \quad (31)$$

It can be noticed that for $n = 2$, (29) coincides only with (30). The inverse QFTs are defined as follows

$$\text{RS – QFT:} \quad h(\boldsymbol{x}) = \int_{\mathrm{R}^2} H_{\mathrm{RS}}(\boldsymbol{f}) \mathrm{e}^{e_2 2\pi f_2 x_2} \, \mathrm{e}^{e_1 2\pi f_1 x_1} d^2 \boldsymbol{x}, \quad (32)$$

$$\text{TS – QFT:} \quad h(\boldsymbol{x}) = \int_{\mathrm{R}^2} \mathrm{e}^{e_1 2\pi f_1 x_1} \, H_{\mathrm{TS}}(\boldsymbol{f}) \mathrm{e}^{e_2 2\pi f_2 x_2} d^2 \boldsymbol{f}. \quad (33)$$

For a 2-D real signal $u(x_1, x_2)$, there exists a formula [9] relating its TS-QFT (30), denoted with U_{TS}, with its Fourier spectrum $U(f_1, f_2)$ as follows

$$U_{\mathrm{TS}}(f_1, f_2) = U(f_1, f_2) \frac{1 - e_3}{2} + U(f_1, -f_2) \frac{1 + e_3}{2}. \quad (34)$$

The above formula proves the equivalence of both representations.

4.2 Hypercomplex 3-D Fourier Transforms

Let us present the definitions (28) and (29) for 3-D signals. The Cayley-Dickson hypercomplex FT is known as the Octonion FT [19] and is denoted with OFT($\boldsymbol{f}$). We have

$$H_{\mathrm{Cl}}(\boldsymbol{f}) = \int_{\mathrm{R}^3} h(\boldsymbol{x}) \mathrm{e}^{-e_1 2\pi f_1 x_1} \, \mathrm{e}^{-e_2 2\pi f_2 x_2} \, \mathrm{e}^{-e_3 2\pi f_3 x_3} d^3 \boldsymbol{x}, \quad (35)$$

$$\mathrm{OFT}(\boldsymbol{f}) = \int_{\mathrm{R}^3} h(\boldsymbol{x}) \mathrm{e}^{-e_1 2\pi f_1 x_1} \, \mathrm{e}^{-e_2 2\pi f_2 x_2} \, \mathrm{e}^{-e_4 2\pi f_3 x_3} d^3 \boldsymbol{x}. \quad (36)$$

Note the different order of imaginary units in (35): e_1, e_2, e_3 in comparison to (36) where we use e_1, e_2, e_4. In consequence, the 3-D hypercomplex analytic signals derived in Clifford and Cayley-Dickson algebras differ. The inverse hypercomplex FTs of (35) and (36) respectively are

$$h(\boldsymbol{x}) = \int_{\mathrm{R}^3} H_{\mathrm{Cl}}(\boldsymbol{f}) \mathrm{e}^{e_3 2\pi f_3 x_3} \, \mathrm{e}^{e_2 2\pi f_2 x_2} \, \mathrm{e}^{e_1 2\pi f_1 x_1} d^3 \boldsymbol{f}, \quad (37)$$

$$h(\boldsymbol{x}) = \int_{\mathrm{R}^3} \mathrm{OFT}(\boldsymbol{f}) \mathrm{e}^{e_4 2\pi f_3 x_3} \, \mathrm{e}^{e_2 2\pi f_2 x_2} \, \mathrm{e}^{e_1 2\pi f_1 x_1} d^3 \boldsymbol{x}. \quad (38)$$

For a 3-D real signal $u(x_1, x_2, x_3)$, the formula relating its OFT (36) and FT - $U(f_1, f_2, f_3)$ has been presented for the first time in [19]. However, we repeated once again the derivation and came to the similar formula differing in signs w.r.t. [19]. The full derivation is presented in the Appendix A. The new formula is

$$\begin{aligned}\mathrm{OFT}(f_1,f_2,f_3) &= \tfrac{1}{4}\left[U(f_3,f_2,f_1)+U(f_3,-f_2,f_1)\right](1-e_5)\\ &+\tfrac{1}{4}\left[U(-f_3,f_2,f_1)+U(-f_3,-f_2,f_1)\right](1+e_5)\\ &+\tfrac{1}{4}e_3\left[U(f_3,f_2,-f_1)-U(f_3,-f_2,-f_1)\right](1+e_5)\\ &+\tfrac{1}{4}e_3\left[U(-f_3,f_2,-f_1)-U(-f_3,-f_2,-f_1)\right](1-e_5).\end{aligned} \tag{39}$$

The relation (39) can be easily verified. So, if we present the spectrum $U(f_1, f_2, f_3)$ as a complex sum of eight terms having different parity w.r.t. a given frequency-domain variable (e - even parity, o – odd parity), i.e.,

$$\begin{aligned}U(f_3,f_2,f_1) &= U_{eee}-U_{eoo}-U_{oeo}-U_{ooe}\\ &+e_1\left(-U_{eeo}-U_{eoe}-U_{oee}+U_{ooo}\right),\end{aligned} \tag{40}$$

and introduce (40) into (39), we come to

$$\begin{aligned}\mathrm{OFT}(f_3,f_2,f_1) &= U_{eee}-e_1U_{eeo}-e_2U_{eoe}+e_3U_{eoo}\\ &-e_4U_{oee}+e_5U_{oeo}+e_6U_{ooe}-e_7U_{ooo}.\end{aligned} \tag{41}$$

We notice that the formula (41) is exactly the same as it has been presented in [19]. The full reasoning is presented in Appendix B.

5. Frequency-Domain Definitions of 2-D and 3-D Analytic Signals

The definitions of complex/hypercomplex FTs recalled in Section 4 can be used to derive the frequency-domain definitions of n-D complex/hypercomplex analytic signals. Calculating the inverse QFT given by (32)-(33) of the 1st-quadrant quaternionic spectra we arrive to (12) and (13). Analogously, introducing into (37)-(38) the octonion spectrum limited to the 1st-octant, we come to definitions (22) and (23).

6. Properties of Hypercomplex Hilbert Spectra in the Cayley-Dickson Algebra

In this section, the properties of complex and hypercomplex Hilbert spectra are studied.

6.1 Relations in 2-D

Let us denote with $V(f_1, f_2)$, $V_1(f_1, f_2)$ and $V_2(f_1, f_2)$ the 2-D Fourier spectra of total and partial Hilbert transforms of $u(x_1, x_2)$ (see (11)). Their relations with the 2-D Fourier spectrum $U(f_1, f_2)$, [15], [16], are:

$$V(f_1,f_2) = -\operatorname{sgn} f_1 \operatorname{sgn} f_2 \cdot U(f_1,f_2), \tag{42}$$

$$V_1(f_1,f_2) = -e_1 \operatorname{sgn} f_1 \cdot U(f_1,f_2), \tag{43}$$

$$V_2(f_1,f_2) = -e_1 \operatorname{sgn} f_2 \cdot U(f_1,f_2). \tag{44}$$

Introducing (42)-(44) into (34), we obtain analogous relations in the quaternion domain:

$$V_\mathrm{q}(f_1,f_2) = (e_1 \operatorname{sgn} f_1)U_\mathrm{q}(f_1,f_2)(e_2 \operatorname{sgn} f_2), \tag{45}$$

$$V_{1\mathrm{q}}(f_1,f_2) = (-e_1 \operatorname{sgn} f_1)U_\mathrm{q}(f_1,f_2), \tag{46}$$

$$V_{2\mathrm{q}}(f_1,f_2) = U_\mathrm{q}(f_1,f_2)(-e_2 \operatorname{sgn} f_2) \tag{47}$$

where $V_\mathrm{q}(f_1, f_2)$, $V_{1\mathrm{q}}(f_1, f_2)$ and $V_{2\mathrm{q}}(f_1, f_2)$ are respectively quaternion spectra of total and partial 2-D Hilbert transforms and $U_\mathrm{q}(f_1, f_2)$ is the quaternion FT of $u(x_1,x_2)$ (see (30)-(31)). Note that the order of multiplication in the above formulas is strictly determined and cannot be changed due to the noncommutativity of the algebra of quaternions.

We now refer to the quaternion analytic signal given by (13). Applying the linearity property of the QFT, we obtain

$$\begin{aligned}&\mathrm{QFT}\{\psi_\mathrm{CD}(\boldsymbol{x})\}\\ &= \mathrm{QFT}\{u\}+\mathrm{QFT}\{e_1v_1\}+\mathrm{QFT}\{e_2v_2\}+\mathrm{QFT}\{e_3v\}.\end{aligned} \tag{48}$$

It has been noticed that if we apply the definition of TS-QFT (31) we arrive to the following relations

$$\mathrm{QFT}_\mathrm{TS}\{e_1v_1\} = e_1\mathrm{QFT}_\mathrm{TS}\{v_1\}, \tag{49}$$

$$\mathrm{QFT}_\mathrm{TS}\{e_2v_2\} = \mathrm{QFT}_\mathrm{TS}\{v_2\}e_2, \tag{50}$$

$$\mathrm{QFT}_\mathrm{TS}\{e_3v\} = e_1\mathrm{QFT}_\mathrm{TS}\{v\}e_2. \tag{51}$$

However using the definition (30), we have

$$\mathrm{QFT}_\mathrm{RS}\{e_1v_1\} = e_1\mathrm{QFT}_\mathrm{RS}\{v_1\}, \tag{52}$$

$$\mathrm{QFT}_\mathrm{RS}\{e_2v_2\} = e_2\mathrm{QFT}_\mathrm{RS}\{v_2\}, \tag{53}$$

$$\mathrm{QFT}_\mathrm{RS}\{e_3v\} = e_3\mathrm{QFT}_\mathrm{RS}\{v\}. \tag{54}$$

As the above relations differ, we see that the choice of the definition of the QFT is not arbitrary.

6.2 Relations in the 3-D Case

The 3-D octonion signal has been defined in (23). In analogy to the 2-D case, its OFT is a sum of OFTs of successive terms. However, the transformation (32) is defined as right-sided and in consequence we obtain the relations analogous to (50)-(54):

$$\mathrm{OFT}\{e_1v_1\} = e_1\mathrm{OFT}\{v_1\}, \tag{55}$$

$$\mathrm{OFT}\{e_2v_2\} = e_2\mathrm{OFT}\{v_2\}, \tag{56}$$

$$\mathrm{OFT}\{e_3 v_{12}\} = e_3 \mathrm{OFT}\{v_{12}\}, \tag{57}$$

$$\mathrm{OFT}\{e_4 v_3\} = e_4 \mathrm{OFT}\{v_3\}, \tag{58}$$

$$\mathrm{OFT}\{e_5 v_{13}\} = e_5 \mathrm{OFT}\{v_{13}\}, \tag{59}$$

$$\mathrm{OFT}\{e_6 v_{23}\} = e_6 \mathrm{OFT}\{v_{23}\}, \tag{60}$$

$$\mathrm{OFT}\{e_7 v\} = e_7 \mathrm{OFT}\{v\}. \tag{61}$$

7. Summary

Two different approaches to the theory of analytic signals have been presented. Their equivalence and mutual relations have been studied. Some new formulas relating hypercomplex Hilbert spectra have been presented. We have presented examples of mesh and contours plots of different parts of the 2-D analytic complex and hyper-complex Cauchy signals. It should be noticed that for 3-D signals, the corresponding functions could be plotted in form of a series of cross-sections with a fixed value of a chosen variable.

Appendix A. Derivation of the Relation (39)

Let us recall the definition of the 3-D FT:

$$U(f_3, f_2, f_1) = \int_{R^3} u(\boldsymbol{x}) \mathrm{e}^{-e_1\alpha_1} \mathrm{e}^{-e_2\alpha_2} \mathrm{e}^{-e_3\alpha_3} d^3\boldsymbol{x} \tag{A1}$$

where $\alpha_i = 2\pi f_i x_i,\ i = 1,2,3$, and $\boldsymbol{x} = (x_3, x_2, x_1)$. Let us calculate the sum

$$\begin{aligned}&\tfrac{1}{2}\left[U(f_3, f_2, f_1) + U(f_3, -f_2, f_1)\right]\\ &\quad = \int_{R^3} u(\boldsymbol{x})\ \mathrm{e}^{-e_1\alpha_1}(\cos\alpha_2)\mathrm{e}^{-e_1\alpha_3} d^3\boldsymbol{x}\end{aligned} \tag{A2}$$

and the difference

$$\begin{aligned}&\tfrac{1}{2}\left[U(f_3, f_2, f_1) - U(f_3, -f_2, f_1)\right]\\ &\quad = \int_{R^3} u(\boldsymbol{x})\ \mathrm{e}^{-e_1\alpha_1}(-e_1\sin\alpha_2)\mathrm{e}^{-e_1\alpha_3}\ d^3\boldsymbol{x}\end{aligned} \tag{A3}$$

Multiplying (A3) from the left by e_3 and applying the multiplication rules of the algebra of quaternions (Tab. 1), we get

$$\begin{aligned}&\tfrac{1}{2}e_3\left[U(f_3, f_2, -f_1) - U(f_3, -f_2, -f_1)\right]\\ &\quad = \int_{R^3} u(\boldsymbol{x})\ \mathrm{e}^{-e_1\alpha_1}(\text{-}e_2\sin\alpha_2)\mathrm{e}^{-e_1\alpha_3} d^3\boldsymbol{x}\end{aligned} \tag{A4}$$

Now adding (A2) and (A4) we obtain

$$\begin{aligned}&\tfrac{1}{2}\left[U(f_3, f_2, f_1) + U(f_3, -f_2, f_1)\right]\\ &\quad + \tfrac{1}{2}e_3\left[U(f_3, f_2, -f_1) - U(f_3, -f_2, -f_1)\right]\\ &\quad = \int_{R^3} u(\boldsymbol{x})\ \mathrm{e}^{-e_1\alpha_1}\mathrm{e}^{-e_2\alpha_2}\mathrm{e}^{-e_1\alpha_3} d^3\boldsymbol{x}\end{aligned} \tag{A5}$$

To simplify the notation, let us introduce

$$\begin{aligned}V(f_3, f_2, f_1) &= \tfrac{1}{2}\left[U(f_3, f_2, f_1) + U(f_3, -f_2, f_1)\right]\\ &\quad + \tfrac{1}{2}e_3\left[U(f_3, f_2, -f_1) - U(f_3, -f_2, -f_1)\right]\end{aligned} \tag{A6}$$

and calculate once again two sums

$$\begin{aligned}&\tfrac{1}{2}\left[V(f_3, f_2, f_1) + V(-f_3, f_2, f_1)\right]\\ &\quad = \int_{R^3} u(\boldsymbol{x})\mathrm{e}^{-e_1\alpha_1}\mathrm{e}^{-e_2\alpha_2}(\cos\alpha_3) d^3\boldsymbol{x},\end{aligned} \tag{A7}$$

$$\begin{aligned}&\tfrac{1}{2}\left[V(f_3, f_2, f_1) - V(-f_3, f_2, f_1)\right]\\ &\quad = \int_{R^3} u(\boldsymbol{x})\mathrm{e}^{-e_1\alpha_1}\mathrm{e}^{-e_2\alpha_2}(-e_1\sin\alpha_3) d^3\boldsymbol{x}\end{aligned} \tag{A8}$$

We notice in (A8) that

$$\begin{aligned}&\tfrac{1}{2}\left[V(f_3, f_2, -f_1) - V(-f_3, f_2, -f_1)\right]\\ &\quad = \int_{R^3} u(\boldsymbol{x})\mathrm{e}^{e_1\alpha_1}\mathrm{e}^{-e_2\alpha_2}(-e_1\sin\alpha_3) d^3\boldsymbol{x}.\end{aligned} \tag{A9}$$

The multiplication of $\mathrm{e}^{e_1\alpha_1}\mathrm{e}^{-e_2\alpha_2}(-e_1\sin\alpha_3)$ from the right by e_5 is equivalent to $\mathrm{e}^{e_1\alpha_1}\mathrm{e}^{e_2\alpha_2}(e_4\sin\alpha_3)$ and in consequence we obtain

$$\begin{aligned}&\tfrac{1}{2}\left[V(-f_3, -f_2, f_1) - V(f_3, -f_2, f_1)\right]e_5\\ &\quad = \int_{R^3} u(\boldsymbol{x})\ \mathrm{e}^{-e_1\alpha_1}\mathrm{e}^{-e_2\alpha_2}(-e_4\sin\alpha_3) d^3\boldsymbol{x}\end{aligned} \tag{A10}$$

Now, we add (A7) and (A10):

$$\begin{aligned}&\tfrac{1}{2}\left[V(f_3, f_2, f_1) + V(-f_3, f_2, f_1)\right]\\ &\quad + \tfrac{1}{2}\left[V(-f_3, -f_2, f_1) - V(f_3, -f_2, f_1)\right]e_5\\ &\quad = \int_{R^3} u(\boldsymbol{x})\ \mathrm{e}^{-e_1\alpha_1}\mathrm{e}^{-e_2\alpha_2}\mathrm{e}^{-e_4\alpha_3}\ d^3\boldsymbol{x}.\end{aligned} \tag{A11}$$

Finally, from (A6) and (A11) we get the formula (39):

$$\begin{aligned}\mathrm{OFT}(f_1, f_2, f_3) &= \tfrac{1}{4}\left[U(f_3, f_2, f_1) + U(f_3, -f_2, f_1)\right](1 - e_5)\\ &\quad + \tfrac{1}{4}\left[U(-f_3, f_2, f_1) + U(-f_3, -f_2, f_1)\right](1 + e_5)\\ &\quad + \tfrac{1}{4}e_3\left[U(f_3, f_2, -f_1) - U(f_3, -f_2, -f_1)\right](1 + e_5)\\ &\quad + \tfrac{1}{4}e_3\left[U(-f_3, f_2, -f_1) - U(-f_3, -f_2, -f_1)\right](1 - e_5).\end{aligned} \tag{A12}$$

×	***1***	***e₁***	***e₂***	***e₃***
1	1	e_1	e_2	e_3
$\boldsymbol{e_1}$	e_1	-1	e_3	$-e_2$
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1
$\boldsymbol{e_3}$	e_3	e_2	$-e_1$	-1

Tab. 1. Multiplication rules in the algebra of quaternions

Appendix B. Proof of the Relation (39)

Let us recall the definition of the 3-D FT given by (40):

$$U(f_3, f_2, f_1) = U_{eee} - U_{eoo} - U_{oeo} - U_{ooe} + e_1(-U_{eeo} - U_{eoe} - U_{oee} + U_{ooo}). \tag{B1}$$

We have the following relations:

$$U(f_3, -f_2, f_1) = U_{eee} + U_{eoo} - U_{oeo} + U_{ooe} + e_1(-U_{eeo} + U_{eoe} - U_{oee} - U_{ooo}), \tag{B2}$$

$$U(-f_3, f_2, f_1) = U_{eee} - U_{eoo} + U_{oeo} + U_{ooe} + e_1(-U_{eeo} - U_{eoe} + U_{oee} - U_{ooo}), \tag{B3}$$

$$U(-f_3, -f_2, f_1) = U_{eee} + U_{eoo} + U_{oeo} - U_{ooe} + e_1(-U_{eeo} + U_{eoe} + U_{oee} + U_{ooo}), \tag{B4}$$

$$U(f_3, f_2, -f_1) = U_{eee} + U_{eoo} + U_{oeo} - U_{ooe} + e_1(U_{eeo} - U_{eoe} - U_{oee} - U_{ooo}), \tag{B5}$$

$$U(f_3, -f_2, -f_1) = U_{eee} - U_{eoo} + U_{oeo} + U_{ooe} + e_1(U_{eeo} + U_{eoe} - U_{oee} + U_{ooo}), \tag{B6}$$

$$U(-f_3, f_2, -f_1) = U_{eee} + U_{eoo} - U_{oeo} + U_{ooe} + e_1(U_{eeo} - U_{eoe} + U_{oee} + U_{ooo}), \tag{B7}$$

$$U(-f_3, -f_2, -f_1) = U_{eee} - U_{eoo} - U_{oeo} - U_{ooe} + e_1(U_{eeo} + U_{eoe} + U_{oee} - U_{ooo}). \tag{B8}$$

Let us introduce (B1)-(B8) into (39) and perform all calculations using the multiplication rules of the algebra of octonions (see Tab. 2). Since this sum is very complicated, we consider each of its components separately. For U_{eee}, we have

$$\tfrac{1}{4}U_{eee}\left[1 - e_5 + 1 + e_5 + e_3(1+e_5) + e_3(1-e_5)\right] = U_{eee}. \tag{B9}$$

Next, we obtain:

$$\tfrac{1}{4}U_{eoo}\left[e_3 \cdot 2 \cdot (1+e_5) + e_3 \cdot 2 \cdot (1-e_5)\right] = e_3 U_{eoo}, \tag{B10}$$

$$\tfrac{1}{4}U_{oeo}\left[e_3 \cdot (-2) \cdot (1-e_5) + e_3 \cdot 2 \cdot (1+e_5)\right] = e_5 U_{oeo}, \tag{B11}$$

$$\tfrac{1}{4}U_{ooe}\left[e_3 \cdot (-2) \cdot (1+e_5) + e_3 \cdot 2 \cdot (1-e_5)\right] = -e_3 e_5 U_{ooe} = e_6 U_{ooe}, \tag{B12}$$

$$\tfrac{1}{4}U_{eeo}\left[e_1 \cdot (-2) \cdot (1-e_5) + e_1 \cdot (-2) \cdot (1+e_5)\right] = -e_1 U_{eeo}, \tag{B13}$$

$$\tfrac{1}{4}U_{eoe}\left[e_3 \cdot (-2e_1) \cdot (1+e_5) + e_3 \cdot (-2e_1) \cdot (1-e_5)\right] = -e_3 e_1 U_{eoe} = -e_2 U_{eoe}, \tag{B14}$$

$$\tfrac{1}{4}U_{oee}\left[(-2e_1) \cdot (1-e_5) + 2e_1 \cdot (1+e_5)\right] = e_1 e_5 U_{oee} = -e_4 U_{oee}, \tag{B15}$$

$$\tfrac{1}{4}U_{ooo}\left[e_3 \cdot (-2e_1) \cdot (1+e_5) + e_3 \cdot 2e_1 \cdot (1-e_5)\right] = -e_3 e_1 e_5 U_{eoe} = -e_7 U_{eoe}. \tag{B16}$$

Finally, we get the definition (41).

×	***1***	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	$\boldsymbol{e_4}$	$\boldsymbol{e_5}$	$\boldsymbol{e_6}$	$\boldsymbol{e_7}$
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\boldsymbol{e_1}$	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
$\boldsymbol{e_3}$	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
$\boldsymbol{e_4}$	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
$\boldsymbol{e_5}$	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
$\boldsymbol{e_6}$	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
$\boldsymbol{e_7}$	e_7	e_6	$-e_5$	$-e_4$	e_3	e_2	e_1	-1

Tab. 2. Multiplication rules in the algebra of octonions

References

[1] SANGWINE, S. J. Fourier transforms of colour images using quaternion or hypercomplex numbers. *Electronics Letters*, 1996, vol. 32, no. 21, p. 1979 - 1980.

[2] BÜLOW, T. Hypercomplex spectral signal representation for the processing and analysis of images. In *Bericht Nr. 99-3*, Kiel (Germany): Institut für Informatik und Praktische Mathematik, Christian-Albrechts-Universität Kiel, 1999.

[3] ELL, T. A., SANGWINE, S. J. Hypercomplex Fourier transforms of color images. *IEEE Transactions on Image Processing*, 2007, vol. 16, no.1, p. 22 - 35.

[4] ALEXIADIS, D. S., SERGIADIS, G. D. Estimation of motions in color image sequences using hypercomplex Fourier transforms. *IEEE Transactions on Image Processing,* 2009, vol. 18, no. 1, p. 168 - 187.

[5] SANGWINE, S. J., ELL, T. A. Color image filters based on hypercomplex convolution. *IEE Proceedings – Vision, Image and Signal processing*, 2000, vol. 147, no. 2, p. 89 - 93.

[6] PEI, S.-C., CHANG, J. H., DING, J.-J. Commutative reduced biquaternions and their Fourier transform for signal and image processing applications. *IEEE Transactions on Signal Process*ing, 2004, vol. 52, no. 7, p. 2012 - 2031.

[7] PEI, S.-C., DING, J.-J., CHANG, J. Color pattern recognition by quaternion correlation. In *Proceedings of the International Conference on Image Process*ing, Thessaloniki (Greece), 2001, p. 894 - 897.

[8] SANGWINE, S. J., ELL, T. A., le BIHAN, N. Fundamental representations and algebraic properties of biquaternions or complexified quaternions. *Advances in Applied Clifford Algebras,* 2011, vol. 21, no. 3, p. 607 - 636.

[9] PEI, S.-C., DING, J.-J., CHANG, J. H. Efficient implementation of quaternion Fourier transform, convolution, and correlation by 2-D complex FFT. *IEEE Transactions on Signal Processing*, 2001, vol. 49, no. 11, p. 2783 - 2797.

[10] HAHN, S. L. The N-dimensional complex delta distribution. *IEEE Transactions on Signal Processing*, 1996, vol. 44, no. 7, p. 1833 - 1837.

[11] DIRAC, P. A. M. *The Principles of Quantum Mechanics*. Oxford (UK): Plenum, 1930.

[12] SCHWARTZ, L., *Méthodes Mathématiques pour les Sciences Physiques*. Paris (France): Hermann, 1965.

[13] SNOPEK, K. M. New hypercomplex analytic signals and Fourier transforms in Cayley-Dickson algebras. *Electronics and Telecommunications Quarterly*, 2009, vol. 55, no. 3, p. 403 - 415.

[14] SOMMER, G. (ed.) *Geometric Computing with Clifford Algebras*. Berlin (Germany): Springer-Verlag, 2001.

[15] HAHN, S. L. Multidimensional complex signals with single-orthant spectra. *Proc. IEEE*, 1992, vol. 80, no. 8, p. 1287 - 1300.

[16] HAHN, S. L. *Hilbert Transforms in Signal Processing*. Artech House, 1996.

[17] BÜLOW, T., SOMMER, G. The hypercomplex signal – a novel extension of the analytic signal to the multidimensional case. *IEEE Transactions on Signal Processing*, 2001, vol. 49, no. 11, p. 2844 - 2852.

[18] HAHN, S. Amplitudes, Phases and Complex Frequencies of 2-D Gaussian Signals. *Bull. Polish Ac. Sci., Tech. Sci.*, 1992, vol. 40, no. 3, p. 289-311.

[19] HAHN S. L., SNOPEK K. M. The unified theory of n-dimensional complex and hypercomplex analytic signals. *Bulletin of the Polish Academy of Sciences, Technical Sciences*, 2011, vol. 59, no. 2, p. 167 - 181.

[20] ELL, T. A. *Hypercomplex Spectral Transforms*. Ph.D. dissertation. Minneapolis (USA): Univ. Minnesota, 1992.

[21] BÜLOW, T., FELSBERG, M., SOMMER, G. Non-commutative hypercomplex fourier transforms of multidimensional signals. In SOMMER, G. (ed.) *Geometric Computing with Clifford Algebras.* Berlin (Germany): Springer, 2001, p. 187 - 229.

About Author …

Kajetana Marta SNOPEK was born in Sosnowiec (Poland) in 1966. She received her M.Sc. in applied mathematics from the Warsaw University of Technology (WUT) in 1991 and the Ph.D. in electronics from the same university in 2002. She is the Assistant Professor at the Institute of Radioelectronics, WUT. Her research interests include signal and system theory, especially multi-dimensional complex and hypercomplex signal processing, time-frequency distributions and their extensions for *n*-D signals.