Closed-Form Approximation for Parallel-Plate Waveguide Coefficients

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Abstract. *Simple closed-form formulas for calculating coefficients of modes excited in a parallel-plate waveguide illuminated by a planar wave are presented. The mode-matching technique and Green's formula are used to arrive at a matrix-based expression for waveguide coefficients calculation. Simplified solution to this matrix is proposed to derive approximate mode coefficient formulas in closed-form for both TE and TM polarization. The results are validated by numerical simulations and show good accuracy for all incidence angles and in broad frequency range.*

Keywords

Mode-matching, radiowave propagation, waveguides.

1. Introduction

Analyses of scattering and transmission of various parallel-plate waveguide structures have spanned across several decades and rely on both analytical and numerical models [1]. A parallel-plate waveguide is a basic building block of more complicated structures with applications of waveguide structure analysis for wireless communications being found in built-up urban [2] and indoor environments [3]. Signal coverage prediction by propagation simulation software requires simple formulas to calculate field distribution in a waveguide that is easily and typically implementable into ray-based, software tools. A relatively simple geometrical problem of a single parallel-plate waveguide has been analytically studied in terms of radiation and transmission properties of this structure [4], [5]. Yet, an accurate analysis of the transmission properties of the parallel-plate waveguide is complicated from a practical point of view. The purpose of this work is to present simple approximations for calculating coefficients of TE and TM modes excited between perfectly conducting plates of a parallel-plate waveguide.

The paper is organized as follows: in Section 2 the problem's geometry is outlined and fields outside and inside the waveguide are defined. In Section 3 both fields are matched at waveguide aperture using Green's theorem and a resulting set of linear equations is solved to obtain coefficient formulas of waveguide modes. Section 4 is focused on validation of the formulas by comparison of the field calculated inside parallel-plate waveguide by several numerical methods.

2. Problem Formulation

The geometry of the problem is shown in Fig. 1 with two perfectly conducting parallel plates forming 'Region II' placed at $x = 0$ and $x = L$. The plates are infinite in the direction of the y axis. The incident plane monochromatic wave, represented by vector $\boldsymbol{k}_{\mathrm{i}}$, propagates from 'Region I' in the x-z plane under angle θ. The θ angle is positive in the case of the x-component of $\boldsymbol{k}_{\mathrm{i}}$ in the positive direction of x axis and negative otherwise.

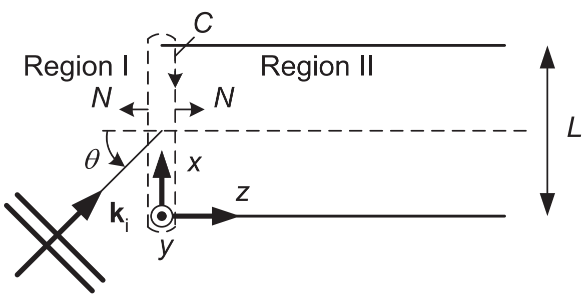

Fig. 1. Parallel plate waveguide geometry.

Waveguide propagation is treated as a two-dimensional problem in the x-z plane. Function ψ represents the y-component of electric or magnetic field intensity with regard to incident TE or TM polarized waves, respectively, and satisfies the two-dimensional Helmholtz equation in the x-z plane

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0. \tag{1}$$

The wave in 'Region I' can be written as the sum of incident and scattered field as [6]

$$\psi^I = e^{-j(\alpha_0 x + \beta_0 z)} + \sum_{m=-\infty}^{\infty} A_m e^{-j(\alpha_m x - \beta_m z)} \tag{2}$$

where unitary amplitude of incident wave is assumed and where α and β are x and z components of wave number k_0, respectively, i.e.

$$a_0 = k_0 sin\theta,$$

$$\beta_0 = \sqrt{{k_0}^2 - {a_0}^2},$$

and

$$a_m = a_0 + \frac{2\pi m}{L},$$

$$\beta_m = \sqrt{{k_0}^2 - {\alpha_m}^2}$$

where $k_0 = 2\pi/\lambda$. The first and second term in (2) represent an incident and scattered wave, respectively. It should be noted that field expansion (2) is used for periodical structures [7]. In this work we use the expansion for single parallel plate waveguide. The representation of the field in 'Region II' for TE and TM polarization in terms of waveguide modes is as follows:

$$\psi^{II}{}_{TE} = \sum_{m=1}^{\infty} D_m sin\left(\frac{m\pi x}{L}\right) e^{-jk_m z}, \quad (3a)$$

$$\psi^{II}{}_{TM} = \sum_{m=0}^{\infty} D_m cos\left(\frac{m\pi x}{L}\right) e^{-jk_m z} \quad (3b)$$

where

$$k_m = \sqrt{{k_0}^2 - \left(\frac{m\pi}{L}\right)^2}$$

where the imaginary part of k_m has to be negative so that the wave amplitude is attenuated along the positive z axis. Only a forward travelling wave in the direction of the z axis is considered in (3a), (3b) neglecting the reflected wave travelling in the opposite direction.

3. Field Matching and Solution

The idea of applying Green's second theorem to relate fields in Regions I and II [6], [8] resides in choosing a suitable integration path to join fields in both regions in one equation. According to Green's second theorem, the following equation holds true for a closed path C around area S

$$\oint_C \left(\phi\frac{\partial\psi}{\partial N} - \psi\frac{\partial\phi}{\partial N}\right) dl = \iint_S (\phi\nabla^2\psi - \psi\nabla^2\phi)\, ds \quad (4)$$

where Φ is an auxiliary function and N is the outer normal of C. If Φ satisfies Helmholtz equation, the right side of (4) is equal to zero. Choosing the integration path C as O-L-L-O as shown in Fig. 1 yields

$$\int_0^L \left(\psi^I\frac{\partial\phi}{\partial z} - \phi\frac{\partial\psi^I}{\partial z}\right) dx + \int_L^0 \left(\psi^{II}\frac{\partial\phi}{\partial z} - \phi\frac{\partial\psi^{II}}{\partial z}\right) dx = 0. \quad (5)$$

Field ψ^I and ψ^{II} in Region I and II is given by (2) and (3a), (3b). Because of an infinite number of coefficients A_m and D_m, an infinite number of pairs of linearly independent equations is needed. It is necessary to solve (5) for a set of linearly independent pairs of auxiliary functions Φ_n. The following pair of auxiliary functions Φ_n was chosen for TE and TM polarization, respectively

$$\phi_n^{+TE} = sin\left(\frac{n\pi x}{L}\right) e^{-jk_n z}, \quad (6a)$$

$$\phi_n^{-TE} = sin\left(\frac{n\pi x}{L}\right) e^{jk_n z}, \quad (6b)$$

$$\phi_n^{+TM} = cos\left(\frac{n\pi x}{L}\right) e^{-jk_n z}, \quad (6c)$$

$$\phi_n^{-TM} = cos\left(\frac{n\pi x}{L}\right) e^{jk_n z} \quad (6d)$$

where n = 1, 2, … in the case of TE polarization and n = 0, 1, … in the case of TM polarization. Inserting Φ_n^+ and Φ_n^- into (5) and performing the integration gives the same pair of linear equations for both TE and TM polarizations.

$$\sum_{m=-\infty}^{\infty} {F_{n,m}}^{TE,TM}(\beta_m + k_n)A_m = {F_{n,0}}^{TE,TM}(\beta_0 - k_n), \quad (7a)$$

$$\sum_{m=-\infty}^{\infty} {F_{n,m}}^{TE,TM}(\beta_m - k_n)A_m + G_n D_n = {F_{n,0}}^{TE,TM}(\beta_0 + k_n) \quad (7b)$$

where

$$G_n = 2Lk_n \qquad \text{for n=0},$$

$$G_n = Lk_n \qquad \text{for n} \neq 0,$$

$${F_{n,m}}^{TE} = \frac{n\pi/L}{(n\pi/L)^2 - {\alpha_m}^2}(1 - cos(n\pi)e^{-j\alpha_m L}), \quad (8a)$$

$${F_{n,m}}^{TM} = \frac{j\alpha_m}{(n\pi/L)^2 - {\alpha_m}^2}(1 - cos(n\pi)e^{-j\alpha_m L}). \quad (8b)$$

Assuming $2M$+1 unknown coefficients A_m, where m = -M, …, -1, 0, 1, … M and corresponding $2M$+1 unknown coefficients D_n, (7a), (7b) give $2M$+1 pairs of equations to solve for $2M$+1 pairs of unknown coefficients A_m and D_n. Although the equations enable us to solve for waveguide coefficients D_n and scattering coefficients A_m, the solution is not given in closed form and requires matrix inversion, or, some other way of obtaining the solution of a set of linear equations. To avoid this inconvenience, simple formulas for waveguide coefficient calculation are proposed.

Considering just the element A_0 of the series expansion in (7a) and setting n = 0 gives

$$A_0 = \frac{\beta_0 - k_0}{\beta_0 + k_0}.$$

Taking just the element A_0 of the series expansion in (7b) and inserting the above formula for A_0 gives

$$D_0 = \frac{4\beta_0 {F_{0,0}}^{TE,TM}}{L}\frac{1}{\beta_0 + k_0}.$$

Repeating the same procedure for n = 1, 2, … while always taking just the element A_0 in the series expansion (7a), (7b) gives the principal result of this study, a general formula for $D_n^{TE,TM}$

$${D_n}^{TE,TM} = \frac{4{F_{n,0}}^{TE,TM}}{L}\frac{{\beta_0}'}{\beta_0 + k_n} \quad (9)$$

where n = 1, 2, … for TE polarization and n = 0, 1, … for TM polarization and where

$$\beta_0{}' = \beta_0/2 \quad \text{for n = 0},$$

$$\beta_0{}' = \beta_0 \quad \text{for n} \neq 0.$$

The obvious simplification in deriving the above TE and TM coefficients resides in omitting all A_m elements in (7a), (7b) except for A_0. This approximation, however, doesn't cause much deviation from accurate results as shown in the next section.

4. Result Validation

In this section, a comparison of field calculations inside the parallel plate waveguide using formula (9) with results obtained by other methods is presented. To validate our results, two approaches were utilized. The first test method is a numerical simulation of the parallel plate waveguide structure using CST Microwave Studio [9]. The second approach is an analytical method for the waveguide mode coefficient calculation using a spectral representation of the field scattered by waveguide by means of the Fourier transform [4]. Although this method is analytical, it requires numerical integration making it more time consuming compared to the proposed solution. The test scenario is a parallel-plate waveguide consisting of two plates 5 cm in length (z-dimension) with a 2-cm gap in between (L = 2 cm). Two-dimensional geometry of the infinite dimension along y-axis is assumed. The point of reception is set at z = 2 cm, x = 1 cm. The result in terms of the y-component of electric and magnetic field E_y and H_y with respect to unitary amplitude of incident electric field is shown in Fig. 2 where $E_y = \psi^{II}{}_{TE}$ and $H_y = \psi^{II}{}_{TM}/Z_0$ where $Z_0 = 120\pi$ is impedance of free space; incident angle θ = 0° is assumed. Field ψ^{II} inside the waveguide was calculated using formulas (3a), (3b) and the proposed simple mode coefficients formulas (9) for TE and TM polarization. For comparison, field calculations using coefficients obtained by matrix inversion of equations (7a,7b) are shown as well. The number of A_m and D_n coefficients is 21. For reference, field calculated using waveguide coefficients obtained by the method based on Fourier transform is plotted. This result, however, is available only for TE polarization. Results of the CST Microwave Studio simulations are shown for both TE and TM polarizations. Similar plots for angle of incidence θ = 30° and θ = 60° are shown in Figs. 3 and 4, respectively. Results show an excellent match between the approach using the set of equations (7a), (7b) and CST simulations. The difference between the results given by the set of linear equations and by the simplified mode coefficients is invariably below 2 dB for all frequencies.

To further investigate the accuracy of the proposed formulas, Figs. 5 and 6 show amplitudes of D_n coefficients as a function of the angle of incidence θ in the same waveguide scenario for both TE and TM polarizations. The amplitudes of the coefficients are normalized with respect to incident field. The frequency of 10 GHz was chosen as the frequency where the highest discrepancies exist between the simple formulas result and other methods in Figs. 2-4. Differences between amplitude coefficients

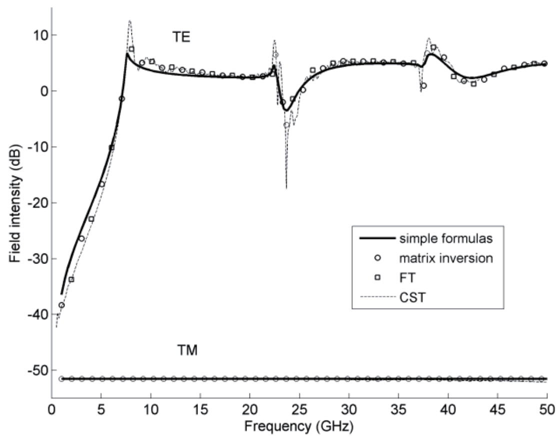

Fig. 2. Comparison of modeling results - incidence angle 0°.

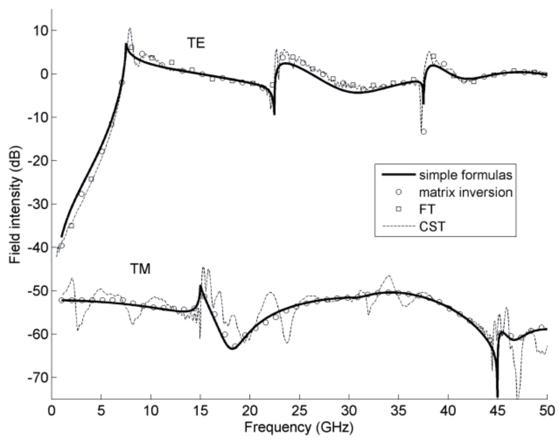

Fig. 3. Comparison of modeling results - incidence angle 30°.

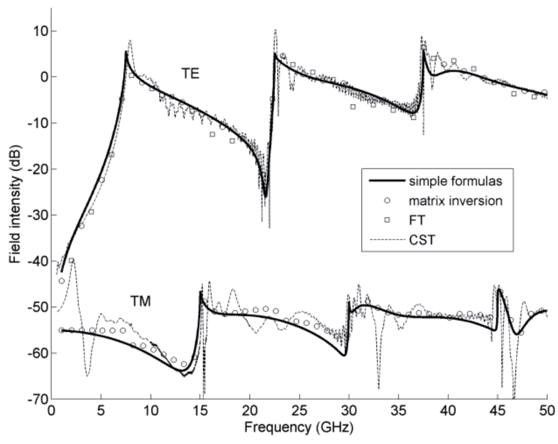

Fig. 4. Comparison of modeling results - incidence angle 60°.

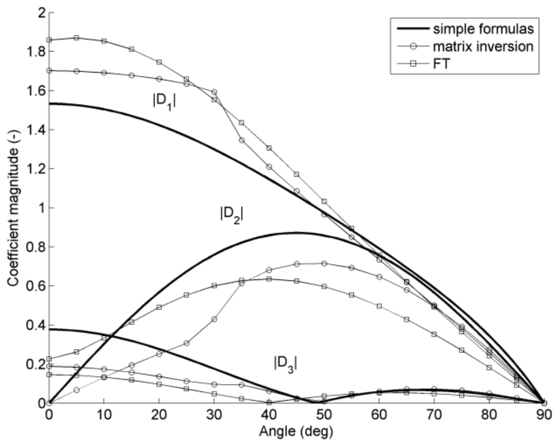

Fig. 5. Calculated coefficient amplitudes, TE polarization.

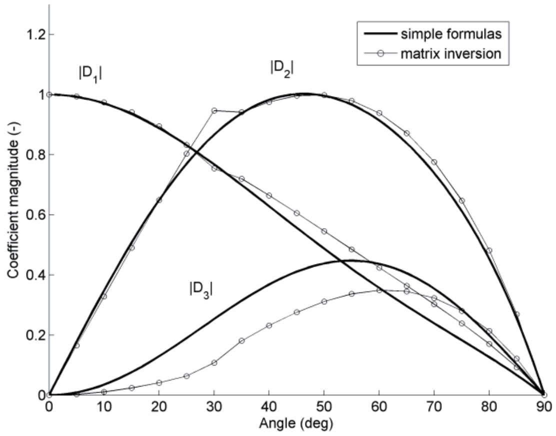

Fig. 6. Calculated coefficient amplitudes, TM polarization.

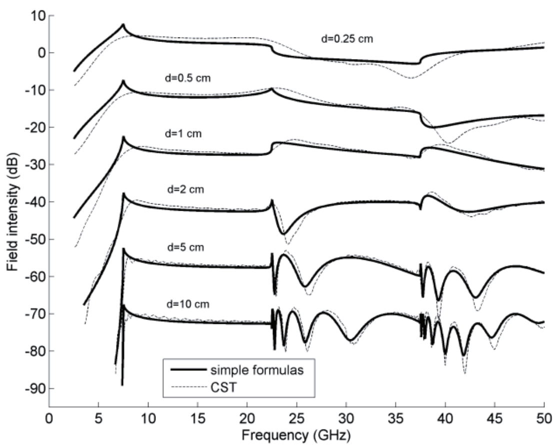

Fig. 7. Waveguide with variable length; TE polarization, incidence angle 0°; *d* is the waveguide length.

calculated by various techniques exist primarily for lower angles of incidence. Validity of the assumption of infinitely long waveguide neglecting reflected wave is illustrated in Fig. 7. The figure shows comparison of CST simulations and the proposed formulas. Separation distance between plates of the waveguide is 2 cm and the field probe is placed at the end of the waveguide; the incident wave is TE polarized with zero incidence angle. The length of the waveguide is changed from 0.25 cm to 10 cm (each subsequent comparison is offset by -15 dB for readability). Good agreement of proposed solution with CST simulation can be observed for long waveguides while for short waveguides the proposed formulas prove to be slightly inaccurate.

5. Conclusion

The mode matching technique was utilized to arrive at a set of linear equations to calculate coefficients of modes excited in a parallel-plate waveguide structure. By simplifying this equation set, simple approximate formulas for mode coefficients were proposed for both TE and TM polarization for an arbitrary angle of incidence. Although the formulas are approximate, results obtained using these simple formulas are well in line with results obtained by other analytical and numerical methods. The proposed formulas represent a reasonable trade-off between accuracy and simplicity of calculation allowing efficient implementation in ray-based propagation prediction tools.

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