All-Pole Recursive Digital Filters Design Based on Ultraspherical Polynomials

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Abstract. *A simple method for approximation of all-pole recursive digital filters, directly in digital domain, is described. Transfer function of these filters, referred to as Ultraspherical filters, is controlled by order of the Ultraspherical polynomial, ν. Parameter ν, restricted to be a non-negative real number ($\nu \geq 0$), controls ripple peaks in the passband of the magnitude response and enables a trade-off between the passband loss and the group delay response of the resulting filter. Chebyshev filters of the first and of the second kind, and also Legendre and Butterworth filters are shown to be special cases of these all-pole recursive digital filters. Closed form equations for the computation of the filter coefficients are provided. The design technique is illustrated with examples.*

Keywords

All-pole IIR filter, lowpass filters, highpass filters, ultraspherical filter, approximation theory.

1. Introduction

The ultraspherical (or Gegenbauer) orthogonal polynomials have already been used in low-pass FIR filter design in time domain [1], [2] and as wavelet functions [3]. However, recursive digital filters can be designed either through application of bilinear transformation on continuous-time filter [4], or directly in the z-domain [5].

In the first approach, the starting point is designing of recursive filters in the continuous-time domain (analog prototype), in addition to designing continuous-time filters based on ultraspherical polynomials [6]. Lastly, transfer function of recursive filter is obtained by using the bilinear transformation. This method requires that all zeros lie at $z = -1$ or on the unit circle.

The second approach is desirable especially for or all-pole (autoregressive) digital filters which have no counterpart in the continuous-time domain. All-pole transfer function class is an important filter category in which low-pass transfer function contains all its zeros at the origin in the z-plane. Those transfer functions are easier to implement than transfer functions that contain only finite zeros on the unit circle, such as elliptic filters.

Discussion in this paper has been restricted to direct design of all-pole digital filters based on ultraspherical polynomials.

Direct design of the recursive digital filters has first been proposed by Rader and Gold [7]. They have shown that characteristic function of these filters is trigonometric polynomial of $\omega/2$, where ω is the digital frequency in radians. They have also concluded that the square of the amplitude characteristic must be rational function of z, where denominator is an image mirror polynomial. Choosing different trigonometric functions for frequency variable, different types of IIR filters can be obtained. Based on these results, direct synthesis of the transitional Butterworth-Chebyshev (TBC) and Butterworth-Legendre filters has been proposed in [8], [9]. These TBC filters are the generalization of the results of previously given continuous-time TBC filters [10], obtained by a mixture of the Butterworth and the Chebyshev components.

Later, other types of orthogonal polynomial approximations for designing continuous-time and IIR digital filters have been used, such as Bessel [11], Jacobi [12], ultraspherical [6] and Pascal polynomials [13]. These approximations are also referred to as polynomial approximations due to the fact that characteristic functions are polynomials. Only Butterworth [7], Chebyshev [14] and transitional Butterworth-Chebyshev [10] continuous-time filters have counterparts in the discrete-time domain.

In this paper a direct method for designing the all-pole recursive digital filters using ultraspherical polynomials, is presented. The frequency responses of ultraspherical filters span between Butterworth to Chebyshev, as the order ν of ultraspherical polynomials goes from infinity to zero. Transition between Butterworth to Chebyshev transfer function is continuous, in contrast with the classical TBC filter where transition is gradual. The order of ultraspherical polynomials, ν, restricted to be a non-negative number, enables a trade-off between the stopband attenuation, passband ripples and group delay deviation of the resulting filter.

The rest of this paper is organized as follows. In Section 2, we derive filter coefficients in closed form and cutoff slope for the proposed design of the all-pole digital filters. Section 3 presents design examples to illustrate the effectiveness of the proposed approach, and finally the conclusions of this paper are presented in Section 4.

2. Approximation

The squared amplitude characteristic of the ultraspherical filters can be expressed as a real function of frequency variable x by using the Feldtkeller's equation [15, Chap. 2]:

$$|H_n(x)|^2 = \frac{1}{1+\varepsilon^2\left[\frac{C_n^\nu(x)}{C_n^\nu(1)}\right]^2} \quad (1)$$

where $C_n^\nu(x)$ is an ultraspherical, also known as Gegenbauer, polynomial (entire even or odd) of order ν (ν is a real number) and degree n. Usually, ε is a design parameter related to the maximum passband attenuation a_{max} (in dB) as $\varepsilon = \sqrt{10^{0.1a_{max}}-1}$.

Formally, ultraspherical polynomials of degree n, $C_n^\nu(x)$, can be defined by the explicit expression [16]:

$$C_n^\nu(x) = \frac{1}{\Gamma(\nu)}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k\Gamma(\nu+n-k)}{k!(n-2k)!}(2x)^{n-2k} \quad (2)$$

or by the recurrence formula:

$$C_n^\nu(x) = \frac{1}{n}[2x(n+\nu-1)C_{n-1}^\nu(x)-(n+2\nu-2)C_{n-2}^\nu(x)] \quad (3)$$

where $C_0^\nu(x)=1$, $C_1^\nu(x)=2\nu x$ and ν acts as a free parameter. Furthermore, $C_n^\nu(x)$ is an even function of x for n even, and odd function of x for n odd. It also has n single zero locations in interval $x\in(-1,1)$.

The ultraspherical polynomials are related to the Chebyshev polynomials of the first kind, $T_n(x)$, to the Legendre polynomials, $P_n(x)$, to the Chebyshev polynomials of the second kind, $U_n(x)$, and to the characteristic polynomial of the Butterworth filter, $B_n(x)$, by following relations [16]:

$$\begin{aligned} T_n(x) &= \frac{n}{2}\lim_{\nu\to 0}\frac{C_n^\nu(x)}{\nu}, \\ P_n(x) &= C_n^{0.5}(x), \\ U_n(x) &= C_n^1(x), \\ B_n(x) &= \lim_{\nu\to\infty}\frac{C_n^\nu(x)}{C_n^\nu(1)} = x^n. \end{aligned} \quad (4)$$

Thus, the ultraspherical responses span between Butterworth to Chebyshev response, as the order ν goes from infinity to zero.

Since, in approximation of all-pole transfer function in (1), term $C_n^\nu(x)=\sum_{i=0}^n g_{n-i}x^{n-i}$ is polynomial, the corresponding squared amplitude characteristic of all pole transfer function takes the following form:

$$|H_n(x)|^2 = \frac{1}{c_{2n}x^{2n}+c_{2n-2}x^{2n-2}+\cdots+c_2x^2+c_0} \quad (5)$$

where

$$c_i = \frac{\varepsilon^2}{(C_n^\nu(1))^2}\sum_{j=0}^{i} g_j g_{i-j}$$

for $i=1,\ldots,2n$ and for $i=0$ holds $c_0=g_0^2+1$ for n even, but $c_0=1$ if n is odd. By convention, $g_{n+1}=\ldots=g_{2n}=0$. Therefore, the magnitude response of all-pole transfer functions (5) is a complete even polynomial.

If x is continuous-time angular frequency $x^2=-s^2$, then function (5) is magnitude characteristic of the continuous time lowpass transfer function. On the other hand, for obtaining the lowpass all-pole discrete-time transfer function a suitable rational function for the frequency variable x is [17]:

$$x^2 = \frac{(z-1)^2}{-4\alpha^2 z} \quad (6)$$

where $\alpha=\sin(\omega_c/2)$ and ω_c is the normalized lowpass cutoff digital frequency in π units. If we want high-pass all-pole filter design, for frequency variable x should be used:

$$x^2 = \frac{(z+1)^2}{4\beta^2 z} \quad (7)$$

where $\beta=\cos(\varpi_c/2)$ and ϖ_c is the normalized highpass cutoff digital frequency in π units. This high-pass approximation is performed by using transformation $z\to -z$ on the lowpass transfer function. The poles of resulting highpass filter are obtained by changing angle by $\pi-\varphi$ where φ is the angle of the lowpass filter pole. This implies that $\omega_c+\varpi_c=\pi$.

Substituting (6) into (5), function $G(z)=H(z)H(1/z)$ is obtained, which is equal to $|H(e^{j\omega})|^2$ when it is evaluated along the unit circle:

$$G(z) = \frac{1}{c_{2n}\frac{(z-1)^{2n}}{(-4\alpha^2 z)^n}+\cdots+c_4\frac{(z-1)^4}{(-4\alpha^2 z)^2}+c_2\frac{(z-1)^2}{-4\alpha^2 z}+c_0}. \quad (8)$$

As can be seen, $G(z)$ is a rational function of z with zero of order n at the origin. Equation (8) can be rewritten in the following form:

$$G(z) = \frac{z^n}{c_{2n}\frac{(z-1)^{2n}}{(-4\alpha^2)^n}+\cdots+c_2\frac{(z-1)^2}{-4\alpha^2}z^{n-1}+c_0z^n}. \quad (9)$$

Note that the component $(z-1)^m$ is a mirror-image polynomial, and that the sum of the mirror-image polynomial of degree m and the mirror-image polynomial of degree $(m-2r)$, multiplied by z^r, is a mirror-image polynomial of degree m. Applying this property, it follows that denominator of $G(z)$ is the mirror-image polynomial of degree $2n$:

$$G(z) = \frac{z^n}{d_0z^{2n}+d_1z^{2n-1}+\cdots+d_nz^n+\cdots+d_1z+d_0}. \quad (10)$$

Relation between coefficients d_i and coefficients c_{2i} is given in closed form by:

$$d_{2n-i} = \sum_{j=0}^{2n-i} \frac{(-1)^j c_{2(i+j-n)}}{(-4\alpha^2)^{i+j-n}} \binom{2(i+j-n)}{j} \tag{11}$$

for $i = n, n+1, \ldots, 2n$.

Poles of the transfer function $H_n(z)$ are obtained by equating the denominator of (10) with zero, and solving it by numerical technique. Since the roots occur in reciprocal pairs, the poles of all-pole ultraspherical filter, $H(z)$, are the roots z_i that lie inside the unit circle:

$$H_n(z) = \frac{h_0 z^n}{\prod_{i=1}^{n}(z - z_i)} = \frac{h_0 z^n}{\sum_{i=0}^{n} a_{n-i}\, z^{n-i}} \tag{12}$$

where $h_0 = \sum_{i=0}^{n} a_i / \sqrt{\sum_{i=0}^{2n} c_i}$ is constant which ensures that amplitude $|H_n(e^{j\omega})|$ is bounded above by unity.

These types of filters can not be obtained from analogue domain by applying the bilinear transformation.

2.1 Cut-off Slope

For filters considered here, a comparison of steepness of their slopes at the cutoff frequency (cutoff slope), can be made by calculating the slopes:

$$S = \frac{\mathrm{d}}{\mathrm{d}\omega} \frac{1}{\sqrt{1 + \varepsilon^2 \left[\frac{C_n^{\nu}(x)}{C_n^{\nu}(1)}\right]^2}} \Bigg|_{\omega=\omega_c} \tag{13}$$

at the cutoff frequency $\omega = \omega_c$ for equal attenuation in the pass-band, a_{max} [6]. Since on the unit circle, $z = \exp(j\omega)$, the frequency variable (6) on the real frequency is:

$$x = \frac{1}{\alpha} \sin \frac{\omega}{2}.$$

By implying the relation [16]:

$$\frac{\mathrm{d}}{\mathrm{d}x} C_n^{\nu}(x) = 2\nu C_{n-1}^{\nu+1}(x)$$

and after simple mathematical manipulation follows:

$$S = -\frac{\varepsilon^2 \nu}{(1+\varepsilon^2)^{3/2}} \frac{C_{n-1}^{\nu+1}(1)}{C_n^{\nu}(1)} \cot \frac{\omega_c}{2}. \tag{14}$$

The cutoff slope depends on the width of the passband, ω_c, and it is steeper if the passband is narrower. When the normalized passband is π, the cutoff slope is equal to zero. In comparison to standard approximation, which uses bilinear transformation [8], this all-pole approximation is suitable for the design of narrow-band lowpass recursive digital filters because it uses $n+1$ multipliers less than for their implementation [18]. For example, if the pass-band edge is less than 0.2π, then both filters have approximately the same slope.

The cutoff slope of highpass all-pole filters depends also on the cutoff frequency:

$$S = \frac{\varepsilon^2 \nu}{(1+\varepsilon^2)^{3/2}} \frac{C_{n-1}^{\nu+1}(1)}{C_n^{\nu}(1)} \tan \frac{\varpi_c}{2}. \tag{15}$$

If cutoff frequency, ϖ_c, increases then cutoff slope also increases. Since the highpass filter has passband above the cutoff frequency, then passband decreases if cutoff frequency increases. In comparison with standard approximation, which uses bilinear transformation, this approximation is suitable also for design narrow-band high pass all-pole digital filter because it saves $(n+1)$ multipliers.

Based on the above-mentioned cutoff slope, cascading low pass filter with high pass filter for the bandpass filter producing is not suitable.

3. Design Examples

Derived equations have been used for calculation of the magnitude and the group delay responses of the Chebyshev of the first and of the second kind, Legendre and Butterworth filters, for degree $n = 8$ and for different values of the parameter ν.

The coefficients of the eight degree transfer functions are given in Tab. 1 ($\nu = 0, 0.5, 1$ and ∞) and corresponding digital frequency responses are displayed in Fig. 1. The frequency is normalized so that the passband edge is $\omega_c = 0.3\pi$ and the maximum passband attenuation is $a_{max} = 2$ dB ($\varepsilon = 0.7647831$).

Coeff.	$A(z) = a_8 z^8 + a_7 z^7 + \cdots + a_1 z + a_0$			
	$\nu = 0$	$\nu = 0.5$	$\nu = 1$	$\nu \to \infty$
a_8	1.000000	1.000000	1.000000	1.000000
a_7	–5.789367	–5.353353	–5.059713	–3.381678
a_6	15.871965	13.635670	12.229774	5.649514
a_5	–26.6694584	–21.321581	–18.172022	–5.830866
a_4	29.891276	22.232672	18.004784	3.988422
a_3	–22.827204	–15.767002	–12.118705	–1.829804
a_2	11.593282	7.411023	5.394609	0.545467
a_1	–3.584770	–2.109682	–1.449659	–0.096038
a_0	0.518558	0.278735	0.179975	0.007612
h_o	0.003399	0.006344	0.009009	0.052630

Tab. 1. Polynomial coefficients for the eight degree ultraspherical filters for different order ν.

When ν is gradually changing from zero to infinity we have a continuous transitional Butterworth-Chebyshev all-pole approximation of recursive digital filters which covers Chebyshev second kind and Legendre approximation. If the degree of the filter is given, transitional region can be continually adjusted with order (ν) of ultraspherical polynomial.

Figure 1 shows the attenuation characteristics of eight degree ultraspherical all-pole filter with $\omega_c = 0.3\pi$ for various values of ν. In Fig. 1 it can be shown that the proposed ultraspherical filter with $\nu > 1$, has very small ripple in the passband and lower group delay variation in comparison to

Chebyshev filter. As might be expected, the Chebyshev filter ($\nu = 0$) has best performance in the stopband. It can be concluded that case $\nu = 1$ is better from the standpoint of amplitude response, but it has a poorer group delay response than the Butterworth filter ($\nu \to \infty$). Order of ultraspherical polynomial, ν, enables trade-off between ripple peaks in passband and delay response of filter.

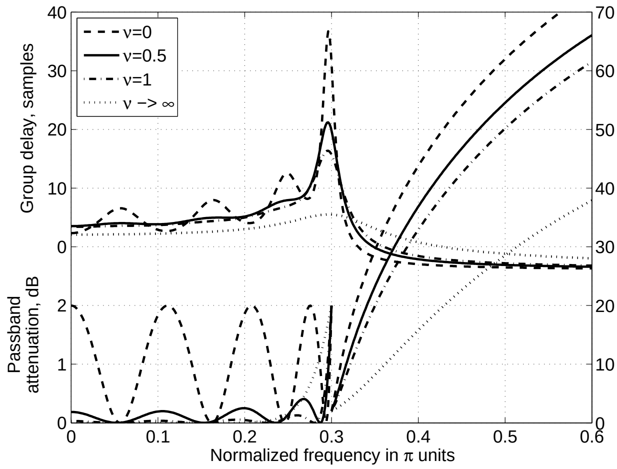

Fig. 1. Attenuation responses and group delay characteristic of the eight-degree ultraspherical all-pole digital filters.

If group distortion is too great, then group delay corrector is available [19].

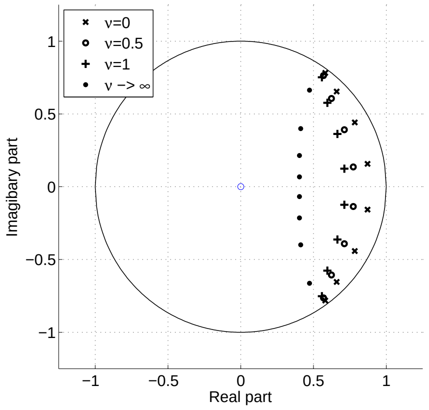

Fig. 2. The pole plot of the eight-degree ultraspherical all-pole digital filters with passband edge $\omega_c = 0.3\pi$.

Figure 2 gives the pole-zero diagram of the eight order ultraspherical recursive digital filters. It is shown that dominant poles of ultraspherical filters for $\nu \leq 1$ are positioned very close to each other, but their dominant pole quality factors (Q-factors) are significantly different.

For example, modulus of dominant poles for thirteenth-degree ultraspherical filters for $\nu = 0$, 0.5 and 1 are 0.98553, 0.97255 and 0.96206, respectively, but their Q-factors[1] are 42.8962708, 22.4701481 and 16.1788120, respectively. As it is known [20, Chapter 5], the sensitivity in the passband increases with pole Q-factor. Thus, the sensitivity in the passband decreases as the order of the ultraspherical polynomial increases.

4. Conclusion

Polynomial approximations, such as Butterworth and Chebyshev, leading to all-pole transfer functions, are extensively used in analog and IIR digital filter design. The Ultraspherical polynomials, $C_n^{\nu}(x)$, are used to present new all-pole IIR discrete-time filter approximation. These filters, which can be referred to as Gegenbauer filters, include as special cases Butterworth ($\nu \to \infty$), Chebyshev second kind ($\nu = 1$), Legendre ($\nu = 0.5$) and Chebyshev first kind ($\nu = 0$) discrete time all-pole filters, amongst others. The order of ultraspherical polynomials, ν, enables a trade-off between the stopband attenuation, the group delay behavior and the passband ripples of the resulting filter. As expected, the group delay becomes more constant as ν deviates from zero (Chebyshev of the first kind) to infinity (Butterworth). The coefficients of the eight order transfer function are tabulated for $\nu = 0, 0.5, 1$ and ∞.

It should be noted that other combinations of ultraspherical polynomials can be used in (1). For example, a product of lower degree ultraspherical polynomials yielding a new one of the same order. Thus, another transfer function is given by

$$|H_n(x)|^2 = \frac{1}{1 + \varepsilon^2 \left[\frac{C_k^{\nu}(x) C_{n-k}^{\nu}(x)}{C_k^{\nu}(1) C_{n-k}^{\nu}(1)}\right]^2} \tag{16}$$

for $k = 0, 1, \ldots, n/2$.

Acknowledgements

This work was supported and funded by the Serbian Ministry of Science and Technological Development under the project No. 32009TR.

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[1] Z-plane may refer to $z = e^{sT}$ or $re^{j\theta} = e^{\sigma T} e^{j\omega}$. Further, $\sigma T = \ln r$ and $\omega = \theta$. Finally,

$$Q = -\frac{\sqrt{\sigma^2 + \omega^2}}{2\sigma} = -\frac{\sqrt{\ln^2 r + \theta^2}}{2 \ln r}.$$

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