Chained-Function Filter Synthesis Based on the Modified Jacobi Polynomials

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Abstract. *A new class of filter functions with pass-band ripple which derives its origin from a method of determining the chained function lowpass filters described by Guglielmi and Connor is introduced. The closed form expressions of the characteristic functions of these filters are derived by using orthogonal Jacobi polynomial. Since the Jacobi polynomials can not be used directly as filtering function, these polynomials have been adapted by using the parity relation for Jacobi polynomials in order to be used as a filter approximating function. The obtained magnitude response of these filters is more general than the magnitude response of published Chebyshev and Legendre chained function filter, because two additional parameters of modified Jacobi polynomials as two additional degrees of freedom are available. It is shown that proposed modified Jacobi chained function filters approximation also includes the Chebyshev chained function filters, the Legendre chained function filter, and many other types of filter approximations, as its special cases.*

Keywords

Chained functions, lowpass filters, modified Jacobi polynomials, return loss, LC ladder network

1. Introduction

It has been about 20 years since the class of filter transfer functions, called chained-function filter, was published by Guglielmi and Conor [1]. With chained-functions, one may define a new polynomial characteristic function that is given by the product of a combination of low degree classical Chebyshev polynomials, called seed functions. The chained function as product of lower degree Legendre polynomials was recently published in the paper [2].

The chained-function concept is provided with a variety of transfer functions, having the same degree but different steady-state response, transient-state response and practical implementation especially with microwave structure characteristics. When compared to the conventional approximation, chained-function concept offers: reduced sensitivity to manufacturing errors, lower resonator unloaded-Q requirements and, consequently, lower filter insertion and return losses. This can be achieved by selecting the appropriate seed function combination for a given implementation on microwave structure technology. This is the results of a tradeoff between passband ripples and out-of-band rejection levels ranging from those associated with Butterworth to the conventional Chebyshev filters.

In this paper, we describe a new family of chained function filters, referred to as the modified Jacobi Chained Function (mJCF) filters. A simple modification of the Jacobi polynomials, which have two parameters, is performed to obtain a new filter approximating function. It is shown that these additional two parameters may be used to obtain a response having either less pass-band ripple level or sharper cutoff than the Chebyshev or Legendre response. The mJCF filter is shown to include as special cases the Chebyshev Chained Function filter, Legendre Chained Function filters, and a filter developed utilizing ultraspherical (Gegenbauer) orthogonal polynomials [3] as seed function. Theoretical and experimental comparisons modified Jacobi chained function filters with the known Chebyshev and Legendre chained function filter's characteristics is not necessary since they are special cases of proposed chained function filters.

1.1 Chained Functions

There are two methods which can be used to synthesize passive filters. First is known as the image parameter method and the second is the insertion-loss method. The first method provides a design that can pass or stop a certain frequency band, but its frequency response cannot be adjusted. The second method is more powerful in the sense that it provides ways to shape the frequency response of the filters.

A chained-function filter design using insertion loss method is presented in the paper. In a doubly terminated two port lossless network driven by voltage source arrangement such as shown in Fig. 1, the power-loss ratio is defined as the ratio of the maximum power available from the source $P_{\max} = V_G^2/(4R_G)$ to the power delivered to the load $P_L = V_o^2/R_L$

$$P_{LR} = \frac{P_{\max}}{P_L} = \frac{R_L}{4R_G}\left|\frac{V_G}{V_O}\right|^2 = \frac{1}{1-|\Gamma(j\omega)|^2} \tag{1}$$

where $|\Gamma(j\omega)|$ is input reflection coefficient. The insertion loss (IL) in dB is $\mathrm{IL} = 10\log_{10} P_{LR}$.

The squared modulus of the transfer coefficient $|H_n(j\omega)|^2 = 1/P_{LR}$ which approximates to the normalized lowpass response can be expressed as follows

$$|H_n(j\omega)|^2 = \frac{1}{1+\varepsilon^2 K_n^2(\omega)} \tag{2}$$

where $\varepsilon < 1$ is a constant, passband edge ripple factor, that controls the power-loss ratio at its pass-band edge, the function $K_n(\omega)$, the characteristic or generating function, is a polynomial of degree n.

For Chained Function filters Guglielmi and Connor [1] have defined new class of characteristic function where $K_n(\omega)$ is given by the product of functions $K_{n_i}(\omega)$, called seed functions, obtaining

$$K_n(\omega) = \prod_{i=1}^{k} K_{n_i}(\omega) \tag{3}$$

and where the overall degree, n, of the filter is given by the sum of the degrees, n_i, of the constituent seed functions: $n = \sum_{i=1}^{k} n_i$. For the seed function, lower degree generalized Chebyshev functions [1] or Zolotarev polynomials [4] can be used, and chained function $K_n(\omega)$ contains transfer function transmission zeros. When all transmission zeros in every seed function approach infinity, the seed functions degenerate to the conventional Chebyshev polynomial [5]. The synthesis of Legendre Chained Function filters was reported in recently published paper [2].

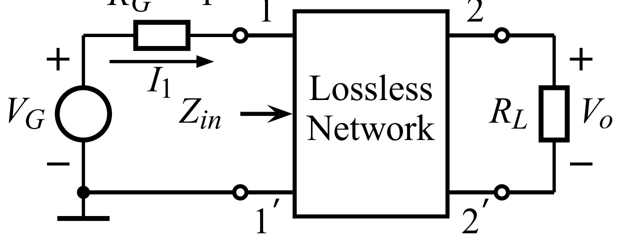

Fig. 1. A doubly terminated two port lossless network driven by voltage source.

2. A Mathematical Background for Modified-Jacobi Polynomials

By using the well known parity relation for classical orthogonal Jacobi polynomials, $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, we present the characteristic of a novel class of lowpass filters, which can be conveniently referred to as the modified Jacobi polynomials, based on the summation of two Jacobi orthogonal polynomials which have the same degree n, as

$$\Pi_n^{(\alpha,\beta)}(x) = \left[P_n^{(\alpha,\beta)}(x) + P_n^{(\beta,\alpha)}(x)\right]\frac{1}{C_n^{(\alpha,\beta)}} \tag{4}$$

where constant $C_n^{(\alpha,\beta)} = P_n^{(\alpha,\beta)}(1) + P_n^{(\beta,\alpha)}(1)$, or in closed form

$$C_n^{(\alpha,\beta)} = \frac{1}{\Gamma(n+1)}\left[\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} + \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)}\right] \tag{5}$$

is chosen in a way that normalization criterion $\Pi_n^{(\alpha,\beta)}(1) = 1$ is satisfied. The resulting modified Jacobi polynomials (4) of degree n are pure odd or pure even polynomials in x, and hence the approximation of the lowpass filters if modified Jacobi polynomial used as characteristic function is possible as in the case of Chebyshev or Legendre polynomials. Modified Jacobi polynomials are symmetrical in relation to the orders α and β, i.e. $\Pi_n^{(\alpha,\beta)}(x) = \Pi_n^{(\beta,\alpha)}(x)$.

In Tab. 1 modified Jacobi polynomials of degrees up to ten for $\alpha = -0.5$ and $\beta = 0.5$, which are generated by the MATLAB symbolic software package, are given.

Plot of the first five modified Jacobi polynomials is illustrated in Fig. 2 for x in $(-1, 1)$ and $n = 0, 1, \ldots, 5$. They satisfy the following relationships: for $|x| < 1$, the characteristic polynomial oscillates around zero and these ripples are bounded by ± 1 for $\alpha, \beta \geq -0.5$. Further, $\Pi_n^{(\alpha,\beta)}(0) \neq 0$ for n even and $\Pi_n^{(\alpha,\beta)}(0) = 0$ for n odd. For $|x| > 1$, the polynomials magnitude increases (decreases) monotonically.

n	$\Pi_n^{(\alpha,\beta)}(x) = [P_n^{(\alpha,\beta)}(x) + P_n^{(\beta,\alpha)}(x)]/C_n^{(\alpha,\beta)}$
0	1
1	x
2	$\frac{4x^2}{3} - \frac{1}{3}$
3	$2x^3 - x$
4	$\frac{16x^4}{5} - \frac{12x^2}{5} + \frac{1}{5}$
5	$\frac{16x^5}{3} - \frac{16x^3}{3} + x$
6	$\frac{64x^6}{7} - \frac{80x^4}{7} + \frac{24x^2}{7} - \frac{1}{7}$
7	$16x^7 - 24x^5 + 10x^3 - x$
8	$\frac{256x^8}{9} - \frac{448x^6}{9} + \frac{80x^4}{3} - \frac{40x^2}{9} + \frac{1}{9}$
9	$\frac{256x^9}{5} - \frac{512x^7}{5} + \frac{336x^5}{5} - 16x^3 + x$
10	$\frac{1024x^{10}}{11} - \frac{2304x^8}{11} + \frac{1792x^6}{11} - \frac{560x^4}{11} + \frac{60x^2}{11} - \frac{1}{11}$

Tab. 1. The modified orthogonal Jacobi polynomials $\Pi_n^{(\alpha,\beta)}(x)$ for $\alpha = -0.5, \beta = 0.5$, and $n = 0, 1, \ldots, 10$.

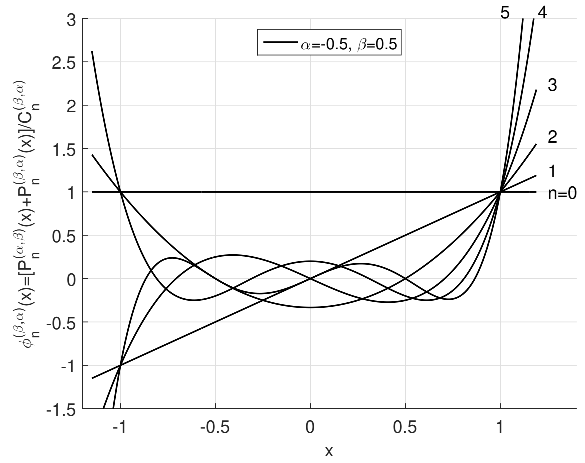

Fig. 2. Plot of the first five modified Jacobi polynomials for $\alpha = -0.5, \beta = 0.5$, and $n = 0, 1, \ldots, 5$.

One can easily show that modified Jacobi polynomial (4) is not orthogonal polynomial except in the case when $\beta = \alpha$ is. For $\beta = \alpha = \lambda$, one can obtain the orthogonal ultraspherical polynomials (symmetric Jacobi polynomials) [6].

Many of the orthogonal polynomials, which are used as filter characteristic function, are special cases of modified Jacobi polynomials. For $\alpha = \beta = \mp 1/2$, the Chebyshev polynomials of first and second kinds, respectively. Further, for $\alpha = \beta = 0$, one can obtain the Legendre polynomials. For the two important special cases $\alpha = -\beta \pm 1/2$, the Chebyshev polynomials of third and fourth kinds are also obtained.

It is important to know where the roots of the modified Jacobi polynomials are located because they have to be in the interval $-1 < x < 1$. After numerical computation, it can be deduced that the modified Jacobi polynomial having degree n has n simple real zeros and $n-1$ local extrema in the open interval $(-1, 1)$. For example, the zeros of the modified Jacobi polynomial of degree 5 with $\alpha = -0.5$ and $\beta = 0.5$ are: $-\sqrt{3}/2$; $-1/2$; 0; $1/2$; $\sqrt{3}/2$. It can be concluded, the zeros of $\Pi_n^{(\alpha,\beta)}(x)$ are located symmetrically about $x = 0$.

Note that modified Jacobi polynomials are the only non orthogonal polynomials which are suitable for the synthesis of the filter function given in a closed form.

3. Filter Transfer Function

The characteristic function, used for the proposed analog filter design, can be either n-the degree modified Jacobi polynomial [7] or a chained-function formed as the product of ν lower degree modified Jacobi polynomials, called modified Jacobi seed functions. Thus, a new family of characteristic, or generating, functions of the same degree n, called modified Jacobi chained-function (mJCF), is given by

$$K_n(\omega) = \prod_{i=1}^{\nu} \Pi_{n_i}^{(\alpha_i,\beta_i)}(\omega) \tag{6}$$

where n_i is the degree of ith seed function ($\nu = 1$ corresponds to the modified Jacobi polynomial), $\Pi_{n_i}^{(\alpha_i,\beta_i)}(\omega)$ is a modified Jacobi polynomial seed function of the degree n_i with parameters α_i and β_i, and n is filter degree. The degree of the filter is given by the sum of the degrees of the constituent seed functions $n = \sum_{i=1}^{\nu} n_i$. All seed functions may have the same, or that each seed function has its own values of the parameters α and β.

The integer partition function $p(n)$, which can be used for determination of number of ways a positive integer n can be written as the sum of positive integers $n_i \le n$, is defined by simple recursion relation [8, page. 825]:

$$p(n) = \sum_{1 \le \frac{k(3k+1)}{2} \le n} (-1)^{k+1} p\Big(n - \frac{3k^2 \pm k}{2}\Big) \tag{7}$$

with $p(0) = 1$ and $p(1) = 1$. Equation (7) provides a simple way to compute the list of values $p(2), p(3), \cdots, p(n-1), p(n)$. As an example, $p(6)$ will be computed using (7) for $k = 1$ and 2 as:

$$\begin{aligned} p(6) &= [p(6-2) + p(6-1)] - [p(6-7) + p(6-5)] \\ &= [5 + 7] - [0 + 1] = 11 \end{aligned}$$

where $p(4) = 5$ and $p(5) = 7$. There are 11 different ways of expressing a seventh-degree modified Jacobi Chained-Function as a product of seed functions of the following degrees: mJCF-6, mJCF-51, mJCF-42, mJCF-411, mJCF-33, mJCF-321, mJCF-3111, mJCF-222, mJCF-2211, mJCF-21111, and mJCF-111111, where each digit represents the degree of constituent seed function n_i. By convention, partitions are usually ordered from largest to smallest. Table 2 shows the resulting 11 mJCF polynomials for $n = 6$ formed with $\alpha = -0.5$ and $\beta = 0.5$.

Seed functions	**mJCF polynomial**
1,1,1,1,1,1	ω^6
2,1,1,1,1	$\frac{4\omega^6}{3} - \frac{\omega^4}{3}$
2,2,1,1	$\frac{16\omega^6}{9} - \frac{8\omega^4}{9} + \frac{\omega^2}{9}$
2,2,2	$\frac{64\omega^6}{27} - \frac{16\omega^4}{9} + \frac{4\omega^2}{9} - \frac{1}{27}$
3,1,1,1	$2\omega^6 - \omega^4$
3,2,1	$\frac{8\omega^6}{3} - 2\omega^4 + \frac{\omega^2}{3}$
3,3	$4\omega^6 - 4\omega^4 + \omega^2$
4,1,1	$\frac{16\omega^6}{5} - \frac{12\omega^4}{5} + \frac{\omega^2}{5}$
4,2	$\frac{64\omega^6}{15} - \frac{64\omega^4}{15} + \frac{16\omega^2}{15} - \frac{1}{15}$
5,1	$\frac{16\omega^6}{3} - \frac{16\omega^4}{3} + \omega^2$
6	$\frac{64\omega^6}{7} - \frac{80\omega^4}{7} + \frac{24\omega^2}{7} - \frac{1}{7}$

Tab. 2. The mJCF polynomials for $n = 6$, $\alpha = -0.5$ and $\beta = 0.5$.

Having established $K_n(\omega)$, and ε, the continuous-time lowpass transfer functions $H_n(s)$ can be found in the usual way: by the analytic continuation of (2) in the whole complex s-plane by putting $\omega = -\mathrm{j}s$, followed by factorisation the denominator, $1 + \varepsilon^2 K_n^2(-\mathrm{j}s)$. The left half of the s-plane factors, $(s - s_i)$, $i = 1, 2, \ldots, n$, correspond to the desired denominator of linear, time invariant, realizable allpole mJCF filter transfer function as

$$H_n(s) = \frac{h_0}{\prod\limits_{i=1}^{n}(s - s_i)} = \frac{h_0}{\sum\limits_{i=1}^{n+1} a_i s^{n-i+1}} = \frac{1}{\sum\limits_{i=1}^{n+1} d_i s^{n-i+1}} \tag{8}$$

where the leading coefficient a_1 is equal to one, $h_0 = a_{n+1}/[1 + \varepsilon^2 K_n^2(0)]^{1/2}$ is just a reference level, i.e. constant that ensures that magnitude $|H_n(\mathrm{j}\omega)|$ is bounded above by unity, and $d_i = a_i/h_0$. If there is at least one of the constituent seed functions that is an odd function, the characteristic function is equal to zero at the zero frequency, i.e. $H_n(0) = 1$, which gives $h_0 = a_{n+1}$ and $d_{n+1} = 1$.

Like the chained function filters based on the Chebyshev and Legendre polynomials, the modified Jacobi chained function filters also have the ripples of the return loss in the region of orthogonality of the Jacobi polynomials, but the values of those ripples and position of return loss zeros in the in-band response can be controlled by parameters α_i and β_i.

Comparison of mJCF filters' cutoff slopes requires calculation of the slope of the magnitude-frequency response function at the passband edge frequency ($\omega = 1$), as obtained from (2) and (6) is equal to

$$S = \frac{\mathrm{d}}{\mathrm{d}\,\omega}|H_n(\mathrm{j}\omega)|\bigg|_{\omega=1} = -\frac{\varepsilon^2}{\left(\varepsilon^2 + 1\right)^{3/2}}\frac{\mathrm{d}}{\mathrm{d}\,\omega}K_n(\omega)\bigg|_{\omega=1} \tag{9}$$

where

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,\omega}K_n(\omega)\bigg|_{\omega=1} &= \sum_{i=1}^{\nu}\frac{\mathrm{d}}{\mathrm{d}\,\omega}\Pi_{n_i}^{(\alpha_i,\beta_i)}(\omega)\bigg|_{\omega=1} \\ &= \sum_{i=1}^{\nu}\frac{n_i(n_i + \alpha_i + \beta_i + 1)\left[\frac{\Gamma(\alpha_i+n_i+1)}{\Gamma(\alpha_i+2)} + \frac{\Gamma(\beta_i+n_i+1)}{\Gamma(\beta_i+2)}\right]}{2\left[\frac{\Gamma(\alpha_i+n_i+1)}{\Gamma(\alpha_i+1)} + \frac{\Gamma(\beta_i+n_i+1)}{\Gamma(\beta_i+1)}\right]}\end{aligned} \tag{10}$$

is the rejection-slope of the characteristic function at cutoff frequency, which is hereafter called the rejection Slope Factor (SF).

It is known that one, and only one, complex pole pair $p_\mathrm{c} = \sigma_\mathrm{c} \pm \mathrm{j}\omega_\mathrm{c}$, whose Q factor $= 0.5\sqrt{\sigma_\mathrm{c}^2 + \omega_\mathrm{c}^2}/\sigma_\mathrm{c}$ is much higher than that of the other pole pairs. This pole pair is called the critical pole pair [9] and its Q factor is denoted by Q_c. The realization of the filters, whose Q_c-factors are high, is difficult and expensive because it is necessary to use components with high stability and to manufacture circuits with good accuracy. In other word, the most sensitive transfer function is the one that has the highest Q_c factor, then it is favourable from an implementation point of view to have as low Q_c factors as possible.

Employing the energy conservation formula for the lossless two-port network given in Fig. 1, $|H_n(\mathrm{j}\omega)|^2 + |\Gamma_n(\mathrm{j}\omega)|^2 = 1$, the magnitude-squared input reflection coefficient can be expressed as:

$$|\Gamma_n(\mathrm{j}\omega)|^2 = \frac{\varepsilon^2 K_n^2(\omega)}{1 + \varepsilon^2 K_n^2(\omega)} \leq 1 \tag{11}$$

where $K_n(\omega)$ is the filter characteristic function given by equation (6). Input return loss in dB defined as $\mathrm{RL} = 20\log_{10}|\Gamma_n(\mathrm{j}\omega)|$ quantifies the amount of impedance matching at the input port. The reflection coefficient is more sensitive to tuning variation and obtaining good return loss provides better system performance and nearly always guarantees good insertion loss result. The maximal return loss level in the passband can be controled with the edge-ripple factor (ε). An insertion loss level of $\mathrm{IL} = 0.0436\,\mathrm{dB}$ at the passband edge frequency ($\varepsilon = 0.1005$) is equivalent to a return loss level of $\mathrm{RL} = -20\,\mathrm{dB}$ at the pass-band edge frequency.

4. Results of Approximation

The preceding information on the filter transfer function $H_n(s)$ is enough for the design of such filters to meet a prescribed cutoff slope and insertion loss as well as return loss tolerances.

An illustration of the proposed approximation in Fig. 3 is given, which shows steady state responses of the three tenth-degree mJCF lowpass filters, mJCF-442, mJCF-811, mJCF-721, with parameters $\alpha_i = -0.5$ and $\beta_i = 0.35$, for $i = 1, 2$ and 3. The denominator coefficients d_i of the low pass prototype transfer function (8) given in Tab. 3, are the counterparts of filters' steady responses given in Fig. 3. The corresponding crtical pole Q factor (Q_c), rejection slope factors (SF) and return loss maximal values (RL) are presented in the last three rows in Tab. 3. Wen the degree of the chained function filter keeps constant and the degree of one seed function increases, then the slope factor and critical pole Q-factor increase. On the other hand, the return loss values can be adjusted with the proper degrees of the seed functions, as it was done for the mJCF-442 in the first column of Tab. 3.

The degree of the constituent seed function has an effect on both the passband response, the rejection-slope and group delay of the resulting mJCF filters. If the degree of one constituent seed function increases and the other decreases, keeping the degree of the filter the same, the maximal passband attenuation, rejection-slope factor and group delay deviation increases.

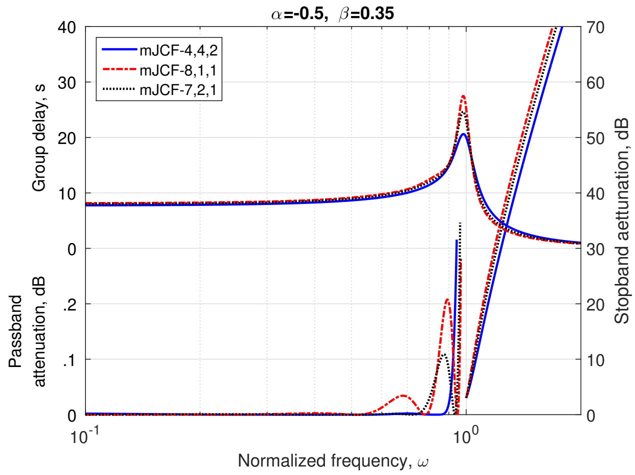

Fig. 3. The frequency responses of the three 10th degree mJCF filters.

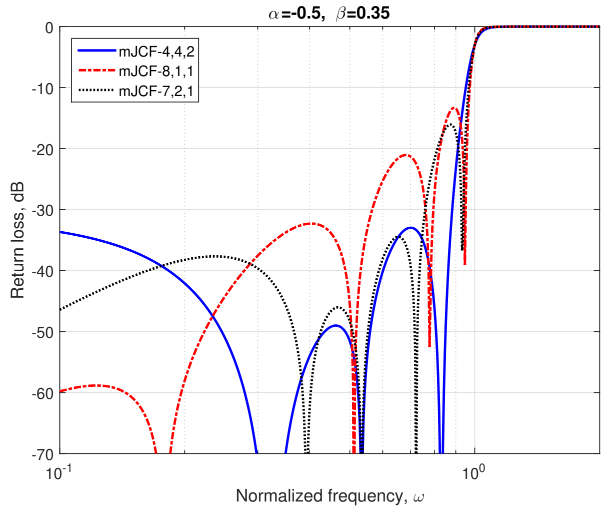

Fig. 4. The return loss characteristic of the three 10th degree mJCF filters.

d_i	The mJCF filter coefficients		
	mJCF-442	mJCF-811	mJCF-721
d_1	18.250855	35.164439	27.159793
d_2	57.957774	96.53012	79.157249
d_3	126.25761	195.77369	165.16355
d_4	189.92166	274.42777	237.7827
d_5	220.95359	300.33103	266.7143
d_6	199.25519	256.3061	232.40188
d_7	140.64633	171.48168	158.80439
d_8	75.835444	87.927797	83.01784
d_9	29.853402	32.925898	31.696307
d_{10}	7.7260096	8.1149119	7.9619479
d_{11}	1.0003661	1.0	1.0
Q_c	6.4084576	9.9818556	8.4768397
SF	20.204406	31.606103	27.048216
RL_{max}	−31.356009	−13.30707	−16.056504

Tab. 3. The coefficients of the denominator polynomials of the three tenth degree mJCF filters with three seed functions with parameters $\alpha = -0.5$ and $\beta = 0.35$ each.

As mentioned earlier, the mJCF filters corresponding to the conventional Chebyshev chained function (CCF) filters if $\alpha = \beta = -0.5$ which have −3dB input return loss ripple level in the passband. In general, this input return loss ripple level is undesirable, but a value less than −20 dB is acceptable in many applications. For this reason, the CCFs require the use the edge ripple factor, ε, for input return loss control.

If $\alpha = -0.5$ and β increases, the ripples in the passband decrease smoothly to be unequal and smaller in magnitude. For $\beta > 1.5$ the passband response is nearly monotonic [7], but the rejection slope factor is much steeper than a Butterworth filter cutoff slope. On the other hand, for $-1 < \beta < -0.5$ the passband ripples are unequal, but in magnitude are larger than 1. These values of β (also for α) have no practical significance. It is shown that the passband ripple can be adjusted to improve the linearity of the phase response at the lower part of the pass band.

In Fig. 4 the input return loss responses of the mentioned three tenth-degree mJCF filters are shown. The return loss of all filters is characterized by a number of lobes. The return loss level of the mJCF-811 and mJCF-721 filters have a maximally value of −13.3071 dB and −16.0565 dB, respectively. Both filters are of odd degree, then the return loss level of such filters is $-\infty$ at the origin. However, the mJCF-442 filter is of even degree, then the return loss at origin has finite value and maximally value of the return loss level is −31.356009 dB.

The frequencies at which the return loss poles of the mJCF-811 and mJCF-721 filters occur are very close to the pass-band edge, but their maximum return loss level is much higher, compared to the mJCF-441 filter.

The most convenient design approach of continuous-time filters, and also microwave filters [10], [11], is to synthesize first a lossless LC ladder lowpass prototype filter having the desired insertion loss response and rejection slope factor. In the case of microwave filters, the second step is to calculate the resonator and coupling parameters that will yield the same insertion loss response on a transformed frequency scale. Return loss (reflection) zeroes can be used for tuning the pass-band response of the filter.

4.1 Doubly Terminated LC Ladder Synthesis

Filters are usually required to have a frequency response with low sensitivity with respect to the non-exact component values. To achieve the lowest sensitivity LC filter network a doubly terminated ladder, as illustrated in Fig. 5, should be used. In this circuit, g_i, $i = 1, 2, \ldots, n$ are LC impedances or admittances (collectively called immittance).

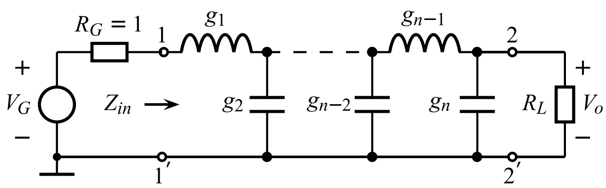

Fig. 5. Doubly terminated low-pass LC ladder network of the even degree.

A powerful method for designing the doubly terminated LC ladder two port networks is the Darlington method [12]. The starting point is to obtain the input driving impedance of the network from a magnitude squared transfer function. Since reflection coefficient $|\Gamma_n(j\omega)|^2$ is known (11), it is possible to determine an appropriate value for $\Gamma_n(s)$ by using the analytic continuation, as in the case of continuous-time transfer function. The poles of the reflection coefficient and of the transfer functions have the same position in the s-plane, but reflection coefficient has the zeros on the imaginary axis. Finally, the reflection coefficient is a rational function of two monic polynomials in s of degree n.

The normalized driving-point impedance $Z_{\text{in}}(s)$ is given by:

$$Z_{\text{in}}(s) = \frac{1 \pm \Gamma_n(s)}{1 \mp \Gamma_n(s)}. \tag{12}$$

Since all the attenuation poles are at infinity, $Z_{\text{in}}(s)$ can be synthesized in the simple LC ladder network.

The equation (12) yields two possible solution for driving point impedance, and both obtained two-port networks are dual to each other. The driving-point impedance (12) is a polynomial rational function of the complex variable s, which can be expanded in a finite continued fraction around $s \to \infty$ starting with the greatest power of s. When the degree of the numerator is greater than the degree of the denominator, the first element's value g_1 which is obtained is a series coil i.e.:

$$\begin{aligned} Z_{\text{in}}(s) &= \frac{N_n(s)}{D_{n-1}(s)} = g_1 s + \frac{R_{n-2}(s)}{D_{n-1}(s)} \\ &= g_1 s + \cfrac{1}{g_2 s + \frac{R_{n-3}(s)}{R_{n-2}(s)}} \end{aligned} \tag{13}$$

where $g_1 s$ and $R_{n-2}(s)$ are the quotient and remainder of the polynomial division $N_n(s)/D_{n-1}(s)$. The degree of $R_{n-2}(s)$ is less than the degree of $D_{n-1}(s)$, therefore the process can be repeated until the remainder g_{n+1} becomes a constant (load resistance R_{L} or conductance $1/R_{\text{L}}$, for n odd or even, respectively).

The lumped element values g_i for the three tenth degree mJCF filters assume to have a cutoff frequency of 1 rad/s and source resistance of 1 Ω, with 3.0103 cutoff point, are shown in the Tab. 4. The eleventh row in the table is the load resistor values R_{L}. For the mLCF-442 the characteristic function (6) is not equal to zero at $\omega = 0$, then the LC ladder network is not terminated with equal resistor, and it is not symmetric and reciprocal. Since the other two ladder networks are odd degree, they are symmetric and reciprocal then less sensitive to the non-exact component values in comparison with the first network.

The ratio of the maximum-to-minimum lumped element values $g_{\max}/g_{\min}$ and the sum of the lumped element values $\sum_{i=1}^{n} g_i$ are commonly used to compare LC ladder networks. In the design of the microwave filters implemented as microstrip stub filters, line width and length increase if element values increase, so it is desirable to keep $g_{\max}/g_{\min}$ and $\sum_{i=1}^{n} g_i$ as small as possible. The element maximum-to-minimum ratio and the sum of filter elements are given in two last row in Tab. 4.

Element	The mJCF filter element's values		
	mJCF-442	mJCF-811	mJCF-721
g_1	0.62979834	0.72856925	0.68622376
g_2	1.4178597	1.6329201	1.5280432
g_3	1.7978409	1.8628213	1.8663292
g_4	1.879938	1.9344207	1.9283815
g_5	2.0290967	1.9561806	1.9529702
g_6	1.922206	1.9561806	1.9529702
g_7	1.9844783	1.9344207	1.9283815
g_8	1.7031325	1.8628213	1.8663292
g_9	1.4967046	1.6329201	1.5280432
g_{10}	0.59662122	0.72856925	0.68622376
$R_{\text{L}} = g_{11}$	1.0556113	1.0	1.0
$g_{\max}/g_{\min}$	2.785043	3.2787647	2.845967
$\sum_{i=1}^{10} g_i$	16.513288	17.229824	16.923896

Tab. 4. Lumped element lowpass lossless LC ladder network resistively terminated at input and output.

Our general opinion is that the mJCF filters could be a suitable candidate for the design of microwave filters. It is known, the most widely used filters in microwave applications are band-pass filters. Using lowpass to bandpass frequency transformation of lumped element lowpass filter, g_i, the series inductor converts to the series resonator and parallel capacitor converts to the parallel resonator. Richards transformation can be used to emulate the inductive and capacitive behaviour of the lumped circuit elements into distributive element consisting the transmission line sections, and Kuroda's identities can be used to facilitate the conversion between the various transmission line realizations.

5. Conclusions

A new class of continuous time allpole filter approximation method, based on utilization of conventional orthogonal Jacobi polynomials, has been presented in this paper. Since the Jacobi polynomial cannot be directly used as filter generating function, a simple modification of Jacobi polynomials is proposed to be use as a generating function. These polynomials, called modified Jacobi polynomials, are very convenient to use for the approximation of the chained function filter, as a seed function. The modified Jacobi function has two free parameters, so that the chained function concept provides a variety of transfer functions, having the same degree, but the different frequency response, transient time response, and implementation characteristics. The Chebyshev chained filter and Legendre chained filters published in research papers are special cases of modified Jacobi chained function filter.

As examples, partitioning three modified Jacobi chained functions into three lower order seed functions, three different responses, having the same degree, pass band and pass band edge ripple factor, but different insertion loss and return loss level, are discussed.

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