History and Progress of Fractional–Order Element Passive Emulators: A Review

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Abstract. This paper presents a state-of-the-art survey in the area of fractional-order element passive emulators adopted in circuits and systems. An overview of the different approximations used to estimate the passive element values by means of rational functions is also discussed. A short comparison table highlights the significance of recent methodologies and their potential for further research. Moreover, the pros and cons in emulation of FOEs are analyzed.

Keywords

Circuit synthesis, constant phase element, fractional-order capacitor, fractional-order element, fractional-order emulator, *RC* network, *RL* network

1. Introduction

The term fractance or fractional-order capacitor (FOC), for an electrical element having properties between those of the resistance and capacitance, was suggested in 1983 by Alain Le Mehaute and Gilles Crepy for denoting electrical elements with non-integer order impedance [1]. In electrical engineering, the constant-phase behavior of capacitors is explained as the frequency dispersion of the capacitance due to the dielectric relaxation, where the electric current density follows the changes in the electric field with a delay. In 1994, Westerlund et al. [2] expressed this phenomenon in real capacitors, in the time domain as:

$$D^{\alpha}v(t) = C^{-1}i(t) \tag{1}$$

where $D^{\alpha}v(t)$ denotes the "fractional-order time derivative". In the same way, the given relationship for fractional-order inductor (FOI) is expressed as:

$$I^{\alpha}v(t) = Li(t) \tag{2}$$

where $I^{\alpha}v(t)$ denotes the "fractional-order time integral" [3] with an order $0 < \alpha < 1$. Figure 1 shows these fundamental components in the frequency domain and possible fractional-order elements (FOEs) in the four quadrants of the complex plane [4]. Their impedance is described as $Z(s) = Ks^{\alpha}$, where ω is the angular frequency with $s = j\omega$ and the phase is given in radians ($\phi = \alpha \pi/2$) or in degrees $(\phi = 90\alpha^{\circ})$. Obviously the impedance of the FOE has a real part dependent on the non-zero frequency and its magnitude varies by 20α dB per decade of frequency. In particular, the impedance of Type IV FOEs, i.e. FOCs in quadrant IV, corresponds to an order $-1 < \alpha < 0$ and a pseudocapacitance $C_{\alpha} = 1/K$, whereas FOIs in quadrant I (Type I) have an order of $0 < \alpha < 1$ and a pseudoinductance of $L_{\alpha} = K$. Their units are expressed as farad·sec^{- α -1} (F·s^{- α -1}) and henry $\sec^{\alpha-1}$ (H·s^{$\alpha-1$}), respectively. Higher order FOCs and FOIs with the described impedances have frequency responses in quadrants II and III (Type II and III), respectively. Their characteristics such as the order (α) , pseudocapacitance (C_{α}) , pseudoinductance (L_{α}) , constant phase zone (CPZ),



Fig. 1. Description of fractional-order elements in the four quadrants of the complex plane [4], [66].



Fig. 2. Progress of fractional-order element emulators.

constant phase angle (CPA - defined phase angle in CPZ), phase angle deviation (PAD - maximum difference between a designed/measured phase and a target phase), or phase band (PB - difference between a designed/measured phase maximum and minimum phase deviation divided by two) influence significantly the transfer function of the fractional systems [5-7]. Therefore, in order to construct fractional operators, finite, infinite, semi-infinite dimensional integerorder systems resulting from the approximation of an irrational function can be used. Indeed, the nature of the fractional differ-integral does not allow the direct implementation in the time-domain simulations of systems with FOEs, while their mathematical representation in the frequency domain leads to irrational functions. Thus, in order to effectively analyze such systems, it is necessary to develop approximations to the fractional operators using standard integer-order operators. These integer-order transfer function approximations then can be used in the design of analog integrator and differentiator circuits by selecting the most adequate distribution of zeros and poles. This strategy makes the task of finding integer-order approximations of fractional transfer functions an important step. Therefore, when models are to be identified, or when simulations are to be performed, fractional transfer functions can be approximated by rational transfer functions, which are easier to handle. A variety of methods for the synthesis of passive FOCs have been proposed that differ from the approximation type. The implementations require standard capacitors and resistors, that are described by conventional (integer) models, but the circuit may include

non-integer order elements. The realization of FOIs using resistive/inductive networks are limited due to their size, cost and limited operating frequency range. Therefore, the research on this area remained limited up to now [4]. In this work, an overview of passive FOE approximation techniques and emulators is provided highlighting the pros and cons of the alternative approaches pointing out the obstacles.

2. Overview of Methods and Structures

In the late 19th century, several researchers worked with the idea of improving the properties of long-distance transmission lines by inserting coils at regular intervals in these lines. Aime Vaschy and Oliver Heaviside were some of those pioneers [8]. The results were not overlooked at that time and no real progress was made until Mihajlo Idvorski Pupin [9] investigated the properties of the transmission cables. Adopting a combination of mathematical and experimental research, Pupin found that the damping in cables for telegraphy and telephony can be substantially reduced by judiciously inserting coils, which resulted in a widespread use of the so-called "Pupin lines". The properties of these lines were further investigated by George Ashley Campbell. In 1903, he published his findings [10] namely that they have a well-defined critical frequency that marks a sudden change in the damping characteristics. While Ashley was investigating these effects, Campbell pointed out that he used this effect to eliminate harmonics in signal generators. In fact, Campbell used the cable as a low-pass filter and moreover, he mentioned the possibility of using the cable as a bandpass filter by replacing the coils by combinations of coils and capacitors. In 1915, Karl Willy Wagner [11], from Germany, and G. A. Campbell, from USA [12], simulated independently the transmission-line by a ladder structure of impedances, mostly constructed as combinations of inductances and capacitances. This invention in 1915 is usually regarded as the birth of an electrical filter. The design theory of this type of filter bears the heritage from the transmissionline theory. Such filters were expressed in terms of characteristic impedances that should be matched if the stages cascaded. Later, from 1930 to 1940, Wilhelm Cauer [13] published a number of articles with the design of passive filters with transfer functions using Chebyshev approximations and having a given attenuation behavior. In 1939, Sidney Darlington [14] followed Cauer with the "insertion-loss theory". Unfortunately, these studies, as they were formulated, had few connection with real-world practice, which made them unpopular. Filters were implemented as networks of inductors, capacitors, and resistors. The problem of bulky and expensive inductors, with low quality factor, directed engineers to use the capacitors in many applications. Moreover, the filter transfer functions obtained with capacitive and resistive elements have their poles on the negative real axis of the Laplace plane. Complex poles are realizable, if active circuits are added. This gave rise to the use of active RC filters [15].

In 1950, S. Darlington [16] proposed a compact form of transfer functions for determining the degree of approximation (n) and for its analysis using pairs of capacitive-resistive phase-shifting networks. In this approach, each network is terminated with its own load along with *n* all-pass sections. Each all-pass sections has the property of exhibiting identical ripple while the frequency range is determined between the lower and upper frequencies, respectively. Therefore, the CPZ is attributed to the phase shift network and is dependent on the complex nature of impedance. A theoretical study showed that the phase error and frequency range are inversely directly proportional to the degree of the approximation, respectively. The main problem of this approximation was the use of inductors and capacitors in all-pass sections, bringing difficulties of the practical circuit realization. In 1959, several networks with a parallel combination of a number of series infinite RC elements to obtain a nearly pure constant argument (phase) over infinite frequency range were proposed by Ralph Morrison [17]. These networks have the basic canonic forms as Foster and Cauer. The constant phase behavior and the scaling factor were described mathematically and their relations were discussed over a two-decade frequency range. The effect of terminating (correcting, single, parallel connected) elements were given and a feedback amplifier design using a constant argument network was explained theoretically.

In 1961, Donald C. Douglas [18] introduced a design procedure to obtain a CPA with a phase from -90° to $+90^{\circ}$ with a predefined error in the specified frequency bandwidth. The phase error is independent from the network complexity, but that does not happen with the frequency range. The theory was based on Morrison's study and explained on a periodic rippling phase function by centering an infinite number of identical characteristics at identical intervals on a logarithmic frequency scale. For a phase angle of 45°, the error was $\pm 0.015^{\circ}$, while the error might vary for other phase angles. Setting the CPA with a given phase error and using simple schematics were the advantages of this method. However, special tables were required to attain the specified phase angle and error. This approach differed from the Morrison's method by the product of an infinite number of basic transfer functions (the Morrison's scheme was based on a summation of an infinite number of basic admittance functions). In 1963, Robert M. Lerner [19] proposed the finite network in which the poles and zeros of the string series of parallel RL and RC pairs were set according to the order k whether equal to be positive or negative. The successive pairs of inductances and capacitances were set in the ratio of $p \in R^+$, and the resistors were in the ratio of p^k . Even though, similar structures were used as in the Morrison's study, Lerner took resistors with values following a power-law instead of p as for the Morrison scheme, thereby reducing the repetitive errors in magnitude. In addition, the compensation impedance was specified to correct the edge ripples that could be modeled as the additional poles in the transfer function. An experimental admittance constructed with five capacitors and five resistors approximated a half-order impedance within the accuracy of 1% in magnitude and $\pm 1^{\circ}$ in phase over the frequency range of 50 Hz-10 kHz.

In 1964, Gordon E. Carlson and Charles A. Halijak [20] showed applications of a Newton process for approximating the characteristics of a balanced symmetric RC lattice (cross RC ladder). The networks were cascades of balanced symmetric lattices with unit resistors in the parallel arms and unit capacitors in the cross arms. The cascade was terminated with a unit resistor. This approximation involved highorder (n^{th}) fractional capacitors whereas iterative methods were not common. The Newton process generated rational functions for the n^{th} root of 1/s. Thus, a fractional capacitor of n^{th} order was formally suggested and investigated the first time in the literature. Until 1967, only a few more studies were held. Kenneth Steiglitz [21] suggested a rational function approximation. Lerner employed a passive building block, and Cassius A. Hesselberth [22] investigated the lumped equivalent of Morrison's RC circuits (Foster-type network). However, these networks were of theoretical interest only, because they were difficult to fabricate with the available technology.

In 1966, Suhash C. Dutta Roy [23] proposed the lumped element model of *RC* networks with an impedance of -45° suitable for fabrication in micro-miniature form using thin

film techniques. Later, he detailed this work [24] on nonuniform networks based on continued fraction expansion (CFE) and compared the results when adopting cascaded networks and rational function approximations. Moreover, the approximation $(1 + s^{\pm 1})^{\pm \lambda}$, where λ is fractional-order and $-1 \le \lambda \le 1$, at low and high frequencies with suitable networks was discussed. Elliptic functions and an equiripple approximation were used. However, the approximations suffered from computational difficulties of the realization complexity.

In 1973, Keith B. Oldham [25] proposed a circuit having two stages, with the resistive-capacitive line subdivided into *n*-equal segments and having each segment replaced by a "T" element composed of two resistors and one capacitor. Then, the geometric ladder was generated by a similar two stage process, being only different from the initial subdivision into unequal segments. The proposed circuits could be adopted for any order, and the resistances and capacitances following a geometric progression leading, therefore, to much simpler calculations. Between 1975 and 1981 [26], six recursive arrangements of RC or RL cells (i.e., a parallel, series and cascade connection of RC and RL cells) were investigated, ensuring non-integer orders not limited to 1/2, being possibly to vary between either 0 and 1, or 0 and -1.

In 1985 V. Ramachandran et al. [27], showed that an irrational immittance of the type $s^{1/2}$ could be realized exactly by a cascade connection of an infinite number of lumped symmetrical two-port networks. The transmission zeros of any two-port can be prespecified, provided they are not located on the negative real axis. Furthermore, the number of transmission zeros of any two-port can be predefined. It was shown that this approach gives a realization where the approximation of $s^{1/2}$ is possible over a large bandwidth.

By 1987 Jia-Chao Wang [28], based on some results of J. Schrama's thesis [29], proposed a systematic way to construct *RC* transmission lines and ladder networks for producing generalized Warburg impedance. The specification of the starting points of the transmission lines was different the one adopted by Schrama. The main outcome was to show a non-constant resistance and capacitance per unit length of the line, while still referring the term "Warburg impedance".

In 1992, Abdelfatah Charef et al. [30], [31] proposed to approximate the fractional power pole. By using a simple graphical method, the zeros and the poles of the approximation for a specified error in dB were found to be in a geometric progression form, which is presently called "Charef's method". In the same year, Masahiro Nakagawa and Kazuyuki Sorimachi proposed a circuit having a fractal structure (tree fractance) composed of resistors and capacitors [32]. The impedance of the element described as $Z(j\omega) = (R/C)^{1/2}\omega^{-1/2} \exp(j\pi/4)$ and the structure was used in fractional integral and differential circuits. In 1993, Khoichi Matsuda [33] presented a design broadband compensator following the H_{∞} control theory. In 1995, Alain Oustaloup [34], [35] developed the so-called Commande Robuste d'Ordre Non Entier (CRONE) suspension from the link between recursive and non-integer derivation. The noninteger derivation using *n* elementary spring-damper cells with time constants distributed recursively, was synthesized based on a pre-defined frequency interval. However, the quality of the Oustaloup's approximation may not be satisfactory in the high and low frequency bands near the fitting frequency bounds. Moreover, it was restricted to odd orders. Later, this problem was solved by Dingyu Xue et al. in 2006 [36].

In 2002, Michio Sugi et al. [37] investigated self-similar ladder circuits with RC elements (domino-type), forming a geometric progression for simulating fractional impedances of various orders. The claimed advantage with these selfsimilar circuits was the characterization of one single optimum pole interval determined by the distributed-relaxationtime models. Up to eighteen sections were used to realize an half-order element within bandwidth of a five decades. In 2005 [38], Yi-Fei Pu et al. [39] analyzed a tree-type network including half-order FOE resembling neural networks and proposed three half-order configurations net-grid type, two-circuit series, and H-type. In 2008, Andrey Arbuzov and Raoul R. Nigmatullin [40] presented three-dimensional selfsimilar RC models of the electric double layer and electrolytic medium that gives fractional impedance response. The complex conductance of circuits are modeled by fractional-power expressions with real and complex-conjugated exponents. Moreover, it was shown that the fractional order was related to the dynamic fractal dimension, providing a geometrical meaning [41] to the concepts.

In 2011, Juraj Valsa et al. [42] proposed a systematic way to simulate the FOE in a desired frequency range for arbitrary orders. The parallel connection of the series of RC elements was used with a parallel, a single resistor and a capacitor as correction elements. The approach was based on a recursive algorithm (RA) allowing the definition of the initial values from commercially available ones. Therefore, the remaining network values were close to the Electronic Industries Alliance (EIA) standard passive elements. By the same year, Dominik Sierociuk et al. [43], [44] introduced a new structure called the nested ladder. Together with the known domino structure, two types of electrical circuits provided the first known examples of circuits, made of passive elements only and exhibiting a behavior of variable order in the time/frequency domain. While the frequency dependent parameter was clear from the Bode plot, the variable order behavior of the circuits in the time domain was designed with help of Mittag-Leffler function as a link between data fitting and fractional-order differential equations [45]. In 2014, Reyad El-Khazali [46], [47] proposed a biquadratic approximation to fractional order differ-integral operators. The circuit was synthesized with series RC and RL networks. The performance of the structures showed better results than equiripple and Oustaloup's approximations, but the obtained passive values were not so realistic.

$C_{\rm p}$ / $R_{\rm p}$	C_1 / R_1	C_2 / R_2	C_3 / R_3	C_4 / R_4	C_5 / R_5	C_6 / R_6	C_7 / R_7
680 pF / 240 kΩ	$820\mathrm{nF}$ / $110\mathrm{k}\Omega$	270 nF / 39 kΩ	91 nF / 15 kΩ	33 nF / 5.1 kΩ	11 nF / 1.8 kΩ	3.9 nF / 680 Ω	1.6 nF / 200 Ω

Tab. 1. EIA standard E24 compliant *RC* values for FOC of order -0.5 and $C_{\alpha_\text{theor}} = 3.75 \,\mu\text{F} \cdot s^{-0.5}$ in CPZ from 1 Hz to 1 MHz via the Valsa network.

In 2019, Avishek Adhikary et al. [48] proposed the design guidelines for Foster-type networks, so that a FOC can be realized with all specified parameters (α , C_{α} , CPZ, CPA, and PAD/PB). The ladder FOC matches the specifications much closer than those obtained by other methods. However, the obtained passive values are still not EIA standard compliant.

Other structures validating the passive and active analog realizations of fractional-order impedances were proposed. We can mention composed of a FOC and some RLC components [49], active cells such as operational amplifiers, operational transconductance amplifiers, current conveyors, and current feedback operational amplifiers [50–52], and several others [53–58].

Evolutionary computing algorithms were used to reduce the drawbacks in optimization methods and to solve complex issues where conventional techniques fail. We find studies in different areas such as in control [59], [60] and chaos [61], or for extracting the design parameters of firstorder high-pass filters [62]. In this regard, we find genetic (GA), cuckoo search, and multi-verse algorithms, and as well as particle swarm, ant-lion, flower pollination, and whale optimizers [62-65]. In the proposed GA algorithm [4], instead of approximating s^{α} using the above approximations at a certain frequency (or bandwidth), a mixed integer-order GA was used for optimizing the phase and magnitude responses of the RC/RL networks in the whole desired frequency range. The proposed GA takes into account all design parameters of the FOEs specified in [48]. However, no correction for using the commercially available RC kit values is needed since the results obtained by the GA directly provides the EIA standard compliant component values as follows. Let us consider a study for comparing the performance of two different approaches. Therefore, for the RC ladder-based FOC design the order ($\alpha = -0.5$), pseudocapacitance ($C_{\alpha_{\text{theor}}} = 3.75 \,\mu\text{F} \cdot s^{-0.5}$), and CPZ (1 Hz to 1 MHz) specifications are the same adopted in [48]. The 9th order admittance function of Foster I [48], and the 8th order admittance Valsa RC function structures considered here are approximated. The distribution of resistance and capacitance values in the RC networks required the FOCs listed in Tab. 1 of this paper and Tab. 5 of [48] are compared in Fig. 3. The passive elements are the EIA standard E24 compliant RC values (5% tolerance) in both cases. We must note that some of the practical values listed in [48] are achieved by a series connection of two resistors or a parallel connection of two capacitors, which increases their complexity. Evaluating the practical values, the total capacitance of GA provided values is about 80% less, while the total resistance is nearly 15%greater than that listed in Tab. 5 of [48]. Figure 3 also indicates that the spread of the components is inferior for the GA



Fig. 3. Distribution of resistance and capacitance values in RC networks realizing FOCs of orders -0.5 and $C_{\alpha_\text{theor}} = 3.75 \,\mu\text{F} \cdot \text{s}^{-0.5}$ in CPZ from 1 Hz to 1 MHz.

Evaluation	[48] Tab. 5	[48] Tab. 5	This Work			
criteria	Theoretical	Practical	Practical			
Used RC network	Fost	Valsa				
Order of approximation	9		8			
α_{mean} (-)	-0.5	-0.5	-0.5			
$C_{\alpha}_{\text{mean}} (\mu \mathbf{F} \cdot s^{-\alpha-1})$	3.86	3.87	3.80			
	Relative magnitude error (%)					
PAD	6.59	8.71	4.76			
Mean	3.01	3.59	1.59			
Median	2.95	3.28	1.42			
Standard Deviation	2.10	2.40	1.09			
	Phase angle error (°)					
PAD	3.47	3.20	1.51			
Mean	1.13	1.21	0.67			
Median	1.17	1.14	0.63			
Standard Deviation	0.63	0.78	0.42			

Tab. 2. Performance characteristics of FOCs of order -0.5 and $C_{\alpha \text{ theor}} = 3.75 \, \mu\text{F} \cdot s^{-0.5}$ in CPZ from 1 Hz to 1 MHz.

Evaluation criteria	[48] Tab. 5 Theoretical	[48] Tab. 5 Practical	This Work Practical
RMSE $ Z (\Omega)$	867.47	1034.85	602.12
RMSE ϕ (°)	1.29	1.44	0.79

Tab. 3. Comparison of the Root Mean Square Errors of the proposed FOCs.

design due to limited range of the values in the population set according to the designer's choice. As a consequence, the mean of the GA values is slightly higher for R and significantly lower for C than those reported in [48]. The ideal and the simulated phase and magnitude responses vs. frequency of theoretical and practical FOCs are depicted in Fig. 4(a) and their performance is reported in Tab. 2. The mean of the fractional order is -0.5 for all realizations in defined CPZ,



Fig. 4. (a) Ideal and simulated phase and magnitude responses vs. frequency, (b) distribution of absolute relative magnitude and phase angle errors of proposed FOCs of orders -0.5.

while the mean of the pseudocapacitance is close to C_{α_theor} . Similarly, the relative magnitude and phase angle errors are the least for this work and their distribution is depicted in Fig. 4(b). The PB values for theoretical/practical [48] and this practical work are $2.78^{\circ}/3.03^{\circ}$ and 1.51° , respectively. Moreover, the accuracy in magnitude and phase of the given FOC realizations, is assessed by means of standard Root Mean Square Error (RMSE). Table 3 again indicates, the GA provided FOC is more precise than the one in [48].

3. Design Boundaries and Constrains

In Sec. 2, various passive electrical networks were discussed for realizing fractional impedances. The irrational impedance is represented commonly in terms of a rational transfer function [6]. Therefore, to implement these functions and to obtain the values of the passive network component, we must have in mind several conditions:

- The transfer function must be real for the Laplace operator *s*.
- The transfer function has distinct features in the complex plane with negative real poles located in the open left-side of *s*-plane.
- The zeros of the input impedance Z(s) pole or the input admittance Y(s) zero should be the closest to the origin of the *s*-plane.

However, validating the conditions formulated above lead also to several drawbacks:

- The approximated rational functions require laborious calculations.
- The constrained optimization to identify the network.
- The passive component values are not optimally scaled and, therefore, negative values may be obtained, which

requires the use of a negative impedance converter.

- The obtained values are not approximated close enough to standardized values, which leads to a poor approximation of the FOE.
- A high number of elements are used to reach a low phase error; hence, the FOE structures must have a large number of branches, that translates to high-order transfer functions.
- The high numbers of elements requires large circuit layouts that result in parasitic effects due to the transmission line particularly at high frequencies.

Therefore, a systematic design procedures should follow the guidelines:

- Use a suitable approximation technique to obtain the impedance in the form Z(s) and to develop it into a suitable expansion.
- Use an appropriate network to reduce the phase ripple at low, mid, and high frequencies. Thus, the network should be selected according to the frequency range of the application.
- Use an evolutionary approximation technique to optimize the design specifications such as constant phase angle, order, pseudocapacitance/pseudoinductance value, phase ripple, constant phase zone, and predefined passive element values.

Fractional-order operations have been approximated also by algorithmic and computational procedures. Digital approximations are necessarily limited in bandwidth, require computer resources, and can suffer from numerical instabilities due to the finite precision arithmetic. These limitations can make digital techniques not the most adequate of solving some problems [49]. Therefore, this type of implementation strategy requires also considerable attention.

4. Conclusions

The recent decades were marked by the development of FOEs and different models of fractal electrodes describing the processes in electrochemical cells. Various approaches to implement models were proposed and studied. However, many aspects are still far from being fully solved. Therefore, the literature survey made in this study help us to trace the chronological evolution of the concepts and approximations. The circuit network realization of FOEs having in mind an optimal design can benefit from artificial neural networks and nature inspired algorithms. The fractional-order, constant phase zone, pseudocapacitance or pseudoinducntance can be set by tuning the circuit parameters e.g., voltage, current, frequency. This can be done by using transistor-based active building blocks and with the conventional IC fabrication technologies. Therefore, this is still an open area and constitutes a promising field for future research.

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