Joint Estimation of Direction and Polarization for Partially Polarized Signals Using Tri-polarized Nested Array

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Abstract. *Using the dual-polarized array for underdetermined estimation of partially polarized (PP) signal parameters can lead to limited signal-to-ratio (SNR) and biased reconstruction of the coherency matrix. In this paper, a new non-iterative method is proposed with the tri-polarized nested array. With the sub-covariance addition, the power of different polarized components of a signal can be completely accumulated, which improves the SNR. Besides, it is proven that with the optimized tri-polarized nested array, noise variance estimation without iterations becomes possible in the underdetermined case, which is critical for unbiased coherency matrix reconstruction. The subspace-based method is adopted to estimate the direction-of-arrival (DOA), and the polarization parameters can be obtained based on the reconstructed coherency matrices. The proposed method is validated by numerical experiments and compared with other representative methods. It has relatively high accuracy and is about one order of magnitude faster than its competitor.*

Keywords

Direction-of-arrival estimation, nested array, polarization estimation, partially polarized signal, tri-polarized sensor

1. Introduction

The DOA estimation using sensor array has been a hot topic of research over the past decades [1]. Apart from DOA, the polarization describing the orientation of the electric field is also an important property of the signal. For example, the polarization information can enrich the signal features and contribute to missions of sensing and recognition. For simultaneously extracting DOAs and polarization parameters from signals, array signal processing with the polarization-sensitive array, which consists of diverse polarized sensors, should be adopted.

With the polarization-sensitive array, many methods for joint estimation of DOA and polarization have been proposed [2–6]. All these methods are based on the assumption that the signals are completely polarized (CP). In fact, the CP signal, which has a fixed polarization state [7], [8], is a limiting case of the more general PP signal whose polarization state can change with time [9]. The PP signal can be expressed as the sum of a CP signal and a randomly polarized signal, and can be found in numerous applications, e.g., radar and ionospheric radio [10]. For the multiple parameters estimation for the PP signals, we also can find some innovative methods in the literature [9, 11–13].

However, the above methods only consider the overdetermined case, where the number of signals is smaller than the number of sensors. For the underdetermined case, where the number of signals is larger than the number of sensors, all the above methods will not work. To resolve more signals than sensors, we need to adopt the sparse array and take advantage of the coarray concept. For the specially designed sparse array, the corresponding coarray has an increased degree of freedom (DOF), which enables the underdetermined parameter estimation. The well-known sparse arrays are the minimum redundant array (MRA) [14], the nested array [15], and the coprime array [16]. Unlike the nested array and coprime array, the MRA can not be designed by an analytical expression. Besides, unlike the nested array and MRA, the coarray of a coprime array is non-consecutive, which usually needs high-complexity algorithms to perform interpolation [17].

Recently, some works which employ the sparse array with dual-polarized sensors to perform parameters estimation for the PP signals are proposed [18–20]. In [18], a set of data correlation sequences is created to take advantage of the increase DOF, and the DOA is estimated by the SS-MUSIC (spatial smoothing-based MUSIC) method [15]. However, no method for polarization estimation is given in this paper. In [19], the quaternion theory is exploited to jointly estimate the DOA and degree of polarization (DOP). Nevertheless, firstly, this method requires that the horizontal and vertical polarization components of a signal have equal power, which

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limits its practical application. Secondly, the estimation of DOP in this method is biased under the underdetermined case since the estimation of noise variance in that situation is not possible. In [20], the DOAs and noise variance are obtained by solving an annihilating relation based optimization problem, and high-accuracy polarization parameters can be estimated owing to the obtained noise variance. However, this method needs iterative calculations, which results in high computational complexity. Besides, the Hermitian property of the coherency matrix has not been exploited during the coherency matrices reconstruction, which further increases the computational complexity. In addition, these three methods only use dual-polarized sensors, which leads to incomplete power accumulation of different polarized components of a signal and a low SNR.

This paper proposes a method for jointly estimating the DOA and polarization parameters that include the DOP, polarization orientation angle (POA), and polarization ellipticity angle (PEA) for PP signals using a nested array with tri-polarized sensors. First, with the sub-covariance addition, the power of different polarized components of a signal can be completely accumulated, which improves the SNR. Besides, it is proven that the noise variance estimation without iterations becomes possible in the underdetermined case, provided the tri-polarized nested array is designed with a given optimization rule. Then, with the estimated noise variance, the unbiased coherency matrix can obtained by a new method which takes the Hermitian property of the coherency matrix into account. The new proposed method does not require iterative calculations. It is validated by numerical experiments and compared with other representative methods, including the Cramer-Rao bound (CRB). The simulation results show that the proposed method possesses high efficiency and relatively high accuracy.

The major contributions of this article are summarized as follows.

1) A non-iterative and accurate method for joint estimation of DOA and polarization is proposed.

2) It is proven that only with the tri-polarized configuration can the noise variance be estimated by the subspace decomposition method under the underdetermined case. This conclusion is critical for proposing the non-iterative method.

3) A rule for optimizing the nested array is proposed, which is used to achieve unbiased reconstructions of the coherency matrices under the underdetermined case and consequently to obtain the unbiased estimation of polarization.

4) To the best of our knowledge, this is the first time that the CRB of tri-polarization configuration for PP signals that is valid under overdetermined and underdetermined cases is derived.

Notations: $(\cdot)^{\mathrm{T}}$, $(\cdot)^{\mathrm{H}}$, $(\cdot)^*$, and $(\cdot)^+$ represent the transpose, conjugate transpose, conjugate, and pseudo-inverse operators, respectively. $\mathbf{I}_L$ denotes the $L \times L$ identity matrix. $\mathrm{vec}(\cdot)$ is the operator that creates a column vector by stacking the column vectors of a matrix below one another. $\mathrm{E}\{\cdot\}$ represents the statistical expectation. $\mathrm{rank}(\cdot)$ means the rank of a matrix. $\mathrm{diag}\{\cdot\}$ represents the operator to form a diagonal matrix with entries of a vector. $\mathrm{Arg}(\cdot)$ takes the principal value of the argument of a complex number. $\|\cdot\|_2$ represents the $\ell 2$ norm. $\otimes$ denotes the Kronecker product. a_i denotes the ith element of vector $\mathbf{a}$. $\mathbf{A}_{\cdot,j}$ represents the j-th column of $\mathbf{A}$, and $[\mathbf{A}]_{i,j}$ represents the element on the i-th row and j-th column of $\mathbf{A}$.

2. Signal Model

Consider the nested array, which is composed of two sub-ULAs. The first one consists of L_1 sensors with inter-element spacing $d = \lambda/2$, and the second one consists of L_2 sensors with inter-element spacing $(L_1 + 1)d$ [15]. λ represents the signal wavelength. According to [15], the number of sensors will be $L = L_1 + L_2$, and the positions of the sensors can be represented by $\boldsymbol{\mu} d$ where

$$\boldsymbol{\mu} = [1, 2, \ldots, L_1, L_1 + 1, 2(L_1 + 1), \ldots, L_2(L_1 + 1)]^{\mathrm{T}}. \quad (1)$$

In order to enable the array to extract polarization information from signals, we apply the sensors which can receive multiple polarized components of signals to the nested array. However, different from the conventional dual-polarized configuration, we propose to adopt the tri-polarized configuration. In the tri-polarized configuration, each sensor, which is called the tri-polarized sensor, has three output ports corresponding to three orthogonal polarization directions, i.e., x, y, and z directions. With the tri-polarized sensors, the conventional nested array becomes the tri-polarized nested array, which is shown in Fig. 1.

Assume K far-field uncorrelated narrowband signals from the direction of $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_K]$ impinge on the array, where $\theta_k \in [0, \pi]$. The vector output of the l-th sensor at snapshot n can be written as

$$\boldsymbol{\eta}_l[n] = \begin{bmatrix} \eta_{l,x}[n] \\ \eta_{l,y}[n] \\ \eta_{l,z}[n] \end{bmatrix} = \sum_{k=1}^{K} a_l(\theta_k)\mathbf{C}_k \mathbf{s}_k[n] + \mathbf{v}_l[n] \quad (2)$$

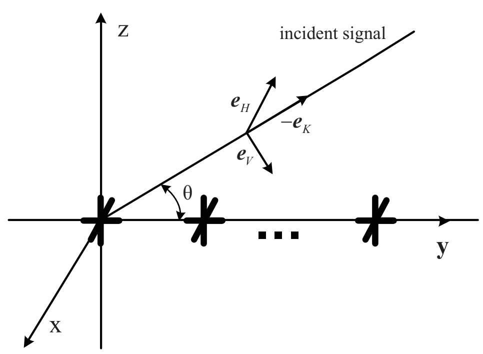

Fig. 1. The tri-polarized nested array.

where $\eta_{l,x}[n]$, $\eta_{l,y}[n]$, and $\eta_{l,z}[n]$ are the outputs corresponding to the x, y, and z polarization directions of the lth sensor, and

$$\mathbf{C}_k = \begin{bmatrix} -1 & 0 \\ 0 & \sin\theta_k \\ 0 & -\cos\theta_k \end{bmatrix} \tag{3}$$

is the polarization response matrix of the k-th signal. $a_l(\theta_k) = \exp(\mathrm{j}2\pi\mu_l d\cos(\theta_k)/\lambda)$ is the spatial response of the k-th signal at the l-th sensor. $\mathbf{s}_k[n] = [s_{k,\mathrm{H}}[n], s_{k,\mathrm{V}}[n]]^{\mathrm{T}}$, where $s_{k,\mathrm{H}}[n]$ and $s_{k,\mathrm{V}}[n]$ are the horizontal and vertical polarization components of the k-th signal, respectively, and the horizontal and vertical directions are reflected by $\mathbf{e}_{\mathrm{H}}$ and $\mathbf{e}_{\mathrm{V}}$ shown in Fig. 1. $\mathbf{v}_l[n] = [v_{l,x}[n], v_{l,y}[n], v_{l,z}[n]]^{\mathrm{T}}$ is the noise vector corresponding to the x, y, and z polarization directions of the l-th sensor. $\mathbf{v}_l[n] \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2\boldsymbol{I}_3)$, where σ_n^2 is the noise variance, is temporally and spatially white and uncorrelated with the signals.

The polarization characteristics of the k-th signal can be captured by the coherency matrix $\mathbf{R}_{s_k}$, which is [13], [21]

$$\begin{aligned}\mathbf{R}_{s_k} &= \mathrm{E}\{\mathbf{s}_k[n]\mathbf{s}_k^{\mathrm{H}}[n]\} = \begin{bmatrix} r_{k,\mathrm{HH}} & r_{k,\mathrm{HV}} \\ r_{k,\mathrm{HV}}^* & r_{k,\mathrm{VV}} \end{bmatrix} \\ &= \frac{\sigma_{k,u}^2}{2}\mathbf{I}_2 + \sigma_{k,c}^2\mathbf{G}(\alpha_k)\mathbf{w}(\beta_k)\mathbf{w}^{\mathrm{H}}(\beta_k)\mathbf{G}^{\mathrm{H}}(\alpha_k)\end{aligned} \tag{4}$$

where $r_{k,\mathrm{HH}}$ and $r_{k,\mathrm{VV}}$ are the power of horizontal and vertical polarization components of the kth signal, respectively, and $r_{k,\mathrm{HV}}$ is the correlation coefficient between $s_{k,\mathrm{H}}[n]$ and $s_{k,\mathrm{V}}[n]$.

$$\mathbf{G}(\alpha_k) = \begin{bmatrix} \cos\alpha_k & \sin\alpha_k \\ -\sin\alpha_k & \cos\alpha_k \end{bmatrix} \tag{5}$$

and

$$\mathbf{w}(\beta_k) = \begin{bmatrix} \cos\beta_k \\ \mathrm{j}\sin\beta_k \end{bmatrix}. \tag{6}$$

As shown in Fig. 2, α_k denotes the POA with $-\pi/2 < \alpha_k \leq \pi/2$, and β_k denotes the PEA with $-\pi/4 \leq \beta_k \leq \pi/4$. $\sigma_{k,c}^2$ and $\sigma_{k,u}^2$ represent the power of polarized component and unpolarized component of the k-th signal, respectively. The DOP of the k-th signal is determined by the formula

$$\rho_k = \sigma_{k,c}^2/(\sigma_{k,u}^2 + \sigma_{k,c}^2). \tag{7}$$

Thus, for a PP signal, we have $\rho_k \in (0,1)$, and its coherency matrix $\mathbf{R}_{\mathbf{s}_k}$ has a full rank.

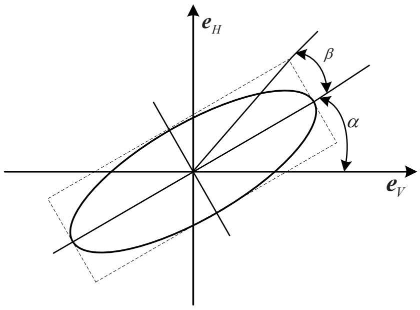

Fig. 2. The polarization ellipse.

Now, the all outputs of the tri-polarized nested array can be arranged in the following compact form

$$\begin{aligned}\boldsymbol{\eta}[n] \in \mathbb{C}^{3L\times 1} &= \left[\boldsymbol{\eta}_l^{\mathrm{T}}[n], \boldsymbol{\eta}_2^{\mathrm{T}}[n], \ldots, \boldsymbol{\eta}_L^{\mathrm{T}}[n]\right]^{\mathrm{T}} \\ &= \mathbf{B}\mathbf{s}[n] + \mathbf{v}[n]\end{aligned} \tag{8}$$

where $\mathbf{B} \in \mathbb{C}^{3L\times 2K} = [\mathbf{a}(\theta_1)\otimes\mathbf{C}_1, \mathbf{a}(\theta_2)\otimes\mathbf{C}_2, \ldots, \mathbf{a}(\theta_K)\otimes\mathbf{C}_K]$, and $\mathbf{a}(\theta_k) = [a_1(\theta_k), a_2(\theta_k), \ldots, a_L(\theta_k)]^{\mathrm{T}}$ is the array steering vector. $\mathbf{s}[n] = \left[\mathbf{s}_1^{\mathrm{T}}[n], \mathbf{s}_2^{\mathrm{T}}[n], \ldots, \mathbf{s}_K^{\mathrm{T}}[n]\right]^{\mathrm{T}}$ and $\boldsymbol{\nu}[n] = \left[\boldsymbol{\nu}_1^{\mathrm{T}}[n], \boldsymbol{\nu}_2^{\mathrm{T}}[n], \ldots, \boldsymbol{\nu}_L^{\mathrm{T}}[n]\right]^{\mathrm{T}}$.

3. The Proposed Method

To make the proposed method easy to follow, we first conclude it in Algorithm 1 and then give the details according to the order of algorithm steps in the following subsections.

Algorithm 1: Joint estimation of direction and polarization for PP Signals using tri-polarized nested array.

Input: The array output $\boldsymbol{\eta}[n], n = 1, 2, \ldots, N$ and number of signals K.

Output: The estimated DOAs and the polarization parameters, including DOPs, POAs, and PEAs.

1 Optimize the nested array via (11).
2 Perform sub-covariance addition via (16).
3 Extract the coarray according to (17).
4 Restore the rank according to (18).
5 Estimate the DOAs via the subspace-based method, e.g. the root-MUSIC in (19) and (20).
6 Reconstruct the coherency matrices according to (23), (24), (26), and (27).
7 Estimate the DOPs, POAs, and PEAs via (28) and (30).

3.1 Optimization of the Nested Array

In Sec. 3.4, we will show that the estimation of the noise variance σ_n^2 is critical for the unbiased reconstructions of the coherency matrices, which will be employed for the estimation of polarization. For the underdetermined estimation of noise variance, we need to optimize the nested array. Before giving the optimization rule, we present the following theorem about the rank of $\mathbf{B}\mathbf{R}_s\mathbf{B}^{\mathrm{H}}$, where $\mathbf{R}_s = \mathrm{E}\{\mathbf{s}[n]\mathbf{s}^{\mathrm{H}}[n]\}$.

Theorem 1. *Assume the PP signals are uncorrelated. Then, for the tri-polarized configuration, the rank of* $\mathbf{B}\mathbf{R}_s\mathbf{B}^{\mathrm{H}}$ *has the following property.*

$$\mathrm{rank}(\mathbf{B}\mathbf{R}_s\mathbf{B}^{\mathrm{H}}) \begin{cases} \leq 2K, & \text{if } K < L \\ \leq L+K, & \text{if } L \leq K < 2L \\ \leq 3L, & \text{if } K \geq 2L \end{cases} \tag{9}$$

Proof. See the detailed proof in Appendix A.

Then, according to (8), the covariance matrix $\mathbf{R}$ can be written as

$$\mathbf{R} = \mathbf{B}\mathbf{R}_s\mathbf{B}^{\mathrm{H}} + \sigma_n^2\mathbf{I}_{3L}. \tag{10}$$

On the basis of Theorem 1, we know, under the overdetermined case, where $K < L$, the dimension of the noise subspace of $\mathbf{R}$ is at least $3L - 2K$. Thus, the estimated noise variance can be obtained by averaging the smallest $3L - 2K$ eigenvalues of $\mathbf{R}$. This method can also be valid under the underdetermined case, where $K \geq L$, by the introduction of an optimization rule to the nested array. As we know, the largest number of resolvable signals of the nested array is $(L_1 + 1)L_2 - 1$. If we set $(L_1 + 1)L_2 - 1 < 2L$, which is equivalent to

$$L_1 < \frac{L_2 + 1}{L_2 - 2} \tag{11}$$

then according to Theorem 1, the dimension of the noise subspace of $\mathbf{R}$ is at least $2L-K$, and the estimated noise variance can be obtained by averaging the smallest $2L-K$ eigenvalues of $\mathbf{R}$. So, we can optimize the nested array according to (11) in order to enable noise variance estimation under the underdetermined case. It needs to be noted that this noise variance estimation method is not valid for the dual-polarized configuration under the underdetermined case. This is because, under that situation, the dimension of the noise subspace will be zero.

3.2 Coarray Construction by Sub-covariance Addition

One of the advantages of the nested array is that a coarray can be constructed from its covariance matrix, and the coarray has larger DOF than the original array, which can improve resolution and enable underdetermined parameter estimation. However, due to the full rank property of the coherency matrix in the model of tri-polarized nested array, direct application of conventional procedure [15] on $\mathbf{R} = \mathrm{E}\{\boldsymbol{\eta}[n]\boldsymbol{\eta}^{\mathrm{H}}[n]\}$ to construct the coarray is infeasible.

To solve this problem, we propose a technique called the sub-covariance addition. Denote the array outputs corresponding to the x, y and z polarization directions by $\mathbf{x}[n]$, $\mathbf{y}[n]$, and $\mathbf{z}[n]$, respectively. Then, we have

$$\begin{cases} \mathbf{x}[n] = \boldsymbol{\eta}_{1:3:3L-2}[n] \\ \mathbf{y}[n] = \boldsymbol{\eta}_{2:3:3L-1}[n] \\ \mathbf{z}[n] = \boldsymbol{\eta}_{3:3:3L}[n] \end{cases} \tag{12}$$

where $\boldsymbol{\eta}_{1:3:3L-2}[n] = [\eta_1[n], \eta_4[n], \ldots, \eta_{3L-2}[n]]^{\mathrm{T}}$, and other expressions are similar. According to (8), the covariance matrices of $\mathbf{x}[n]$, $\mathbf{y}[n]$, and $\mathbf{z}[n]$ can be expressed respectively as

$$\mathbf{R}_{xx} = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k) r_{k,\mathrm{HH}} + \sigma_n^2\mathbf{I}_L, \tag{13}$$

$$\mathbf{R}_{yy} = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k) r_{k,\mathrm{VV}} \sin^2\theta_k + \sigma_n^2\mathbf{I}_L, \tag{14}$$

$$\mathbf{R}_{zz} = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k) r_{k,\mathrm{VV}} \cos^2\theta_k + \sigma_n^2\mathbf{I}_L. \tag{15}$$

Then, by adding these three covariance matrices, we obtain a new covariance matrix, which is

$$\begin{aligned} \mathbf{R}' &= \mathbf{R}_{xx} + \mathbf{R}_{yy} + \mathbf{R}_{zz} \\ &= \sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k) p_k + 3\sigma_n^2\mathbf{I}_L \end{aligned} \tag{16}$$

where $p_k = r_{k,\mathrm{VV}} + r_{k,\mathrm{HH}} = \sigma_{k,u}^2 + \sigma_{k,c}^2$. Since $\mathbf{R}_{xx}$, $\mathbf{R}_{yy}$ and $\mathbf{R}_{zz}$ are all the sub-covariance matrices of $\mathbf{R}$, we call (16) the sub-covariance addition.

From (16), two points need to be stated. The first is that (16) is similar to the covariance matrix of the conventional nested array model, which means the coarray can be constructed by conventional procedure. The second is that k-th signal in (16) is the sum of the powers of horizontal and vertical polarization components. Thus, the powers of different polarized components of a signal are completely accumulated. By contrast, in the dual-polarized configuration like [18] and [19], the powers of different polarized components of a signal can at most be partially accumulated.

Then, by vectorizing $\mathbf{R}'$ and averaging the elements corresponding to the same lags, we can obtain the coarray outputs $\boldsymbol{\gamma}$. According to the theory of nested array [15], $\boldsymbol{\gamma}$ can be written as

$$\boldsymbol{\gamma} \in \mathbb{C}^{(2L_2(L_1+1)-1)\times 1} = \mathbf{A}'\mathbf{p} + 3\sigma_n^2\mathbf{i}. \tag{17}$$

Since the coarray of the nested array is a virtual uniform linear array (ULA), $\mathbf{A}'$ is the manifold matrix of a ULA with $2L_2(L_1 + 1) - 1$ elements and d inter-element spacing. $\mathbf{p} = [p_1, p_2, \ldots, p_K]^{\mathrm{T}}$, and $\mathbf{i}$ is a vector whose middle element is 1 with the remaining elements being 0.

3.3 Toeplitz Matrix for Rank Restoration

It is worth noting that (17) is a single measurement vector (SMV) model, leading to $\mathrm{rank}(\mathrm{E}\{\boldsymbol{\gamma}\boldsymbol{\gamma}^{\mathrm{H}}\}) = 1$. The rank deficiency problem makes it difficult to estimate parameters of multiple signals. The classical solution is the spatial smoothing technique, which has been adopted in [18]. However, this method requires some matrices addition. Here, we replace the spatial smoothing with Toeplitz construction which requires no matrices addition. Especially when the number of sensors is large, the computational complexity of Toeplitz construction will be far less than that of spatial smoothing. By using the Toeplitz construction method, we can obtain a new covariance matrix $\mathbf{R}_{\mathrm{T}}$ which is expressed as [22]

$$\mathbf{R}_{\mathrm{T}} = \begin{bmatrix} \gamma_{L_2(L_1+1)} & \gamma_{L_2(L_1+1)-1} & \cdots & \gamma_1 \\ \gamma_{L_2(L_1+1)+1} & \gamma_{L_2(L_1+1)} & \cdots & \gamma_2 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{2L_2(L_1+1)-1} & \gamma_{2L_2(L_1+1)-2} & \cdots & \gamma_{L_2(L_1+1)} \end{bmatrix}. \tag{18}$$

It can be proven that when $L_2(L_1 + 1) > K$, we have $\mathrm{rank}(\mathbf{R}_{\mathrm{T}}) > K$, and the signal subspace of $\mathbf{R}_{\mathrm{T}}$ is the same as the column space of $\mathbf{V} = [\mathbf{v}(\theta_1), \mathbf{v}(\theta_2), \ldots, \mathbf{v}(\theta_K)]$, where $v_l(\theta_k) = \exp(\mathrm{j}2\pi(l-1)d\cos(\theta_k)/\lambda)$, $l = 1, 2, \ldots, L_2(L_1+1)$.

3.4 Direction and Polarization Estimation

For the algorithm realization, all the covariance matrices are estimated by their sample covariance matrices. For example, $\mathbf{R}_{xx}$ in (13) is estimated by $\mathbf{R}_{xx} = \sum_{n=1}^{N} \mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n]/N$, where N is the number of snapshots.

1) Direction Estimation

Since the rank of $\mathbf{R}_{\mathrm{T}}$ is larger than K, we can adopt the subspace-based method, e.g., MUSIC, root-MUSIC, or ESPRIT, to perform DOA estimation. Take the root-MUSIC method for instance. Perform eigendecomposition on $\mathbf{R}_{\mathrm{T}}$ to extract the noise subspace $\mathbf{U}_n$ which is spanned by the eigenvectors corresponding to the smallest $L_2(L_1+1)/2-K$ eigenvalues, and then construct the following polynomial [23]

$$f(\upsilon) = \boldsymbol{\upsilon}'^{\mathrm{T}}\mathbf{U}_n\mathbf{U}_n^{\mathrm{H}}\boldsymbol{\upsilon}^{\mathrm{T}} \tag{19}$$

where $\boldsymbol{\upsilon}' = [1, \upsilon^{-1}, \upsilon^{-2}, \ldots, \upsilon^{-L_2(L_1+1)+1}]^{\mathrm{T}}$, and $\boldsymbol{\upsilon} = [1, \upsilon, \upsilon^2, \ldots, \upsilon^{L_2(L_1+1)-1}]^{\mathrm{T}}$. Denote its K largest roots inside the unit circle by $\upsilon_1, \upsilon_2, \ldots, \upsilon_K$. The DOAs can be estimated as

$$\hat{\theta}_k = \arccos\left[\frac{\angle(\upsilon_k)\lambda}{2\pi d}\right], \quad k = 1, 2, \ldots, K \tag{20}$$

where $\angle(\cdot)$ is used to take the principal argument of a complex number.

2) Polarization Estimation

To estimate the polarization, we first need to reconstruct the coherency matrices $\mathbf{R}_{s_k}, k = 1, 2, \ldots, K$. In other words, we first need to estimate $r_{k,\mathrm{HH}}$, $r_{k,\mathrm{VV}}$, and $r_{k,\mathrm{HV}}, k = 1, 2, \ldots, K$.

For the estimation of $r_{k,\mathrm{HH}}$, vectorizing $\mathbf{R}_{xx}$ and substituting $\hat{\theta}_k$ into (13), we have

$$\mathrm{vec}(\mathbf{R}_{xx} - \sigma_n^2\mathbf{I}_L) = \hat{\mathbf{A}}_1\mathbf{r}_{\mathrm{HH}} \tag{21}$$

where $\mathbf{r}_{\mathrm{HH}} = [r_{1,\mathrm{HH}}, \ldots, r_{K,\mathrm{HH}}]^{\mathrm{T}}$, and $\hat{\mathbf{A}}_1 = [\mathbf{a}^*(\hat{\theta}_1) \otimes \mathbf{a}(\hat{\theta}_1), \ldots, \mathbf{a}^*(\hat{\theta}_K) \otimes \mathbf{a}(\hat{\theta}_K)]$. For the estimation of $r_{k,\mathrm{VV}}$, we additionally introduce $\mathbf{R}_{yz} = \mathrm{E}\{\mathbf{y}[n]\mathbf{z}^{\mathrm{H}}[n]\} = -\sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k)r_{k,\mathrm{VV}}\sin(\theta_k)\cos(\theta_k)$. Then vectorizing $\mathbf{R}_{yy}$, $\mathbf{R}_{zz}$, and $\mathbf{R}_{yz}$ and replacing θ_k with $\hat{\theta}_k$, we have

$$\begin{bmatrix} \mathrm{vec}(\mathbf{R}_{yy} - \sigma_n^2\mathbf{I}_L) \\ \mathrm{vec}(\mathbf{R}_{zz} - \sigma_n^2\mathbf{I}_L) \\ \mathrm{vec}(\mathbf{R}_{yz}) \end{bmatrix} = \hat{\mathbf{A}}_2\mathbf{r}_{\mathrm{VV}} \tag{22}$$

where $\mathbf{r}_{\mathrm{VV}} = [r_{1,\mathrm{VV}}, \ldots, r_{K,\mathrm{VV}}]^{\mathrm{T}}$, and

$$[\hat{\mathbf{A}}_2]_{\cdot,k} = \begin{bmatrix} \sin^2\hat{\theta}_k \\ \cos^2\hat{\theta}_k \\ -\sin(\theta_k)\cos(\theta_k) \end{bmatrix} \otimes \left(\mathbf{a}^*(\hat{\theta}_k) \otimes \mathbf{a}(\hat{\theta}_k)\right).$$

As shown in (21) and (22), we still need to estimate the noise variance σ_n^2 in order to achieve unbiased estimations of the coherency matrices. According to Sec. 3.1, the estimated noise variance $\hat{\sigma}_n^2$ can be obtained by averaging the smallest $2L-K$ eigenvalues of $\mathbf{R}$ in the underdetermined case or by averaging the smallest $3L-2K$ eigenvalues of $\mathbf{R}$ in the overdetermined case. Then, substituting $\hat{\sigma}^2$ into (21) and (22), we can estimate $r_{k,\mathrm{HH}}$ and $r_{k,\mathrm{VV}}, k = 1, 2, \ldots, K$ by

$$\hat{\mathbf{r}}_{\mathrm{HH}} = \hat{\mathbf{A}}_1^{+}\mathrm{vec}(\mathbf{R}_{xx} - \hat{\sigma}_n^2\mathbf{I}_L) \tag{23}$$

and

$$\hat{\mathbf{r}}_{\mathrm{VV}} = \hat{\mathbf{A}}_2^{+}\begin{bmatrix} \mathrm{vec}(\mathbf{R}_{yy} - \hat{\sigma}_n^2\mathbf{I}_L) \\ \mathrm{vec}(\mathbf{R}_{zz} - \hat{\sigma}_n^2\mathbf{I}_L) \\ \mathrm{vec}(\mathbf{R}_{yz}) \end{bmatrix}, \tag{24}$$

respectively.

For the estimation of $r_{k,\mathrm{HV}}$, we introduce $\mathbf{R}_{xy} = \mathrm{E}\{\mathbf{x}[n]\mathbf{y}^{\mathrm{H}}[n]\} = -\sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k)r_{k,\mathrm{HV}}\sin\theta_k$ and $\mathbf{R}_{xz} = \mathrm{E}\{\mathbf{x}[n]\mathbf{z}^{\mathrm{H}}[n]\} = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{a}^{\mathrm{H}}(\theta_k)r_{k,\mathrm{HV}}\cos\theta_k$. Then we can derive the following equation

$$\mathrm{vec}(\mathbf{R}_{xz}) - \mathrm{vec}(\mathrm{j}\mathbf{R}_{xy}) = \hat{\mathbf{A}}_1\hat{\boldsymbol{\Phi}}\mathbf{r}_{\mathrm{HV}} \tag{25}$$

where $\hat{\boldsymbol{\Phi}} = \mathrm{diag}\{[\exp(\mathrm{j}\hat{\theta}_1), \ldots, \exp(\mathrm{j}\hat{\theta}_K)]^{\mathrm{T}}\}$, and $\mathbf{r}_{\mathrm{HV}} = [r_{1,\mathrm{HV}}, \ldots, r_{K,\mathrm{HV}}]^{\mathrm{T}}$. Then $r_{k,\mathrm{HV}}, k = 1, 2, \ldots, K$ can be estimated by

$$\hat{\mathbf{r}}_{\mathrm{HV}} = (\hat{\mathbf{A}}_1\hat{\boldsymbol{\Phi}})^{+}\left[\mathrm{vec}(\mathbf{R}_{xz}) - \mathrm{vec}(\mathrm{j}\mathbf{R}_{xy})\right]. \tag{26}$$

Then, the K coherency matrices can be reconstructed as

$$\begin{bmatrix} \mathrm{vec}(\hat{\mathbf{R}}_{s_1}) \\ \mathrm{vec}(\hat{\mathbf{R}}_{s_2}) \\ \vdots \\ \mathrm{vec}(\hat{\mathbf{R}}_{s_K}) \end{bmatrix} = \mathrm{vec}\left([\hat{\mathbf{r}}_{\mathrm{HH}}, \hat{\mathbf{r}}_{\mathrm{HV}}^*, \hat{\mathbf{r}}_{\mathrm{HV}}, \hat{\mathbf{r}}_{\mathrm{VV}}]^{\mathrm{T}}\right) \tag{27}$$

where the Hermitian property of the coherency matrix is exploited.

At last, the polarization parameters of the k-th signal can be estimated based on the reconstructed coherency matrix $\hat{\mathbf{R}}_{s_k}, k = 1, 2, \ldots, K$. Assume that the two eigenvalues of the matrix $\hat{\mathbf{R}}_{s_k}$ are $u_{k,1}$ and $u_{k,2}$ that satisfy $u_{k,1} > u_{k,2}$, and the eigenvector corresponding to $u_{k,2}$ is $\boldsymbol{\xi}_k = [\xi_{k,1}, \xi_{k,2}]^{\mathrm{T}}$. Then, according to the definition of coherency matrix in (4), the DOP of the k-th signal can be estimated as

$$\hat{\rho}_k = \frac{u_{k,1} - u_{k,2}}{u_{k,1} + u_{k,1}}. \tag{28}$$

Moreover, on the basis of subspace orthogonality, it is easy to find that the POA and PEA of the k-th signal can be estimated by solving the following equations

$$[\mathbf{G}(\alpha_k)\mathbf{w}(\beta_k)]^{\mathrm{H}}\boldsymbol{\xi}_k = \mathbf{0} \tag{29}$$

and the solutions are

$$\hat{\alpha}_k = \mathrm{Arg}(\omega)/2, \quad \hat{\beta}_k = \pi/4 - \arctan|\omega| \tag{30}$$

where

$$\omega = \frac{\mathrm{j}\xi_{k,2} - \xi_{k,1}}{\mathrm{j}\xi_{k,2} + \xi_{k,1}}. \tag{31}$$

The proof of (30) can be found in Appendix B.

Parameter	Signal 1	Signal 2	Signal 3	Signal 4	Signal 5	Signal 6	Signal 7	Signal 8
DOA	30.00°	47.14°	64.29°	81.43°	98.57°	115.71°	132.86°	150.00°
DOP	0.90	0.90	0.90	0.90	0.90	0.90	0.90	0.90
POA	−60.00°	−42.86°	−25.71°	−8.57°	8.57°	25.71°	42.86°	60.00°
PEA	−36.00°	−25.71°	−15.43°	−5.14°	5.14°	15.43°	25.71°	36.00°

Tab. 1. The parameters of the PP signals.

3.5 Computational Complexity Analysis

In Algorithm 1, the most time-consuming steps are step 5, step 6, and step 7. In step 5, if the root-MUSIC method is used, the operations of eigendecomposition and polynomial rooting requires $O((L_2(L_1+1))^3)$ and $O((2L_2(L_1+1)-1)^3)$, respectively. In step 6, the most time-consuming operations are the calculations of (23), (24), and (26), which cost $O(K^2(L^2+K))$, $O(K^2(3L^2+K))$, and $O(K^2(L^2+K))$, respectively. In step 7, the most time-consuming operations are the eigendecomposition of K 2-by-2 matrices, which requires $O(8K)$ in total. The computational complexity of the proposed method is also reflected by the algorithm runtime in the simulation part.

4. Simulation Results

In this section, we evaluate the performance of the proposed method by numerical simulations. The layout of the nested array is set as $L_1 = L_2 = 3$, which satisfies the optimization rule in (11). Thus, the number of physical sensors is $L = L_1 + L_2 = 6$. We consider the underdetermined case, which is more challenging than the overdetermined case. Assume there are eight signals, which is larger than the number of sensors. The DOAs, DOPs, POAs, and PEAs used in all simualtions are listed in Tab. 1. For performance comparison, the method in [20] labeled as ANN, the method in [18] labeled as SS-MUSIC, and the method in [19] labeled as Quaternion are chosen. The CRB of the estimation whose derivation is shown in Appendix C is also included as the performance reference. The simulation are run by Matlab 2021a on a computer equipped with AMD 3900x CPU and 64GB RAM. The code of the proposed method will be avaiable at https://github.com/pyj8711/EfficientJDP upon acceptance of the paper.

In the first simulation, we show the validity of the proposed method. The SNR is set as 10 dB, and the number of snapshots is set as 500. The proposed method is trialed 10 times on independent and identically distributed samples. The estimations of DOAs, DOPs, POAs, and PEAs for eight signals in ten trials are shown in Fig. 3. It can be found from Fig. 3 that all the estimated DOAs and polarization parameters are close to their true values.

In the second simulation, the proposed method and the other methods are compared with respect to the DOA estimation performance. The performance is evaluated by root-mean-square error (RMSE) of DOA estimation, which is defined by

$$\mathrm{RMSE}_{\mathrm{DOA}} = \sqrt{\frac{1}{Q}\sum_{q=1}^{Q}\frac{\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}^{(q)}\|_2^2}{K}} \tag{32}$$

where $\hat{\boldsymbol{\theta}}^{(q)}$ contains the estimated DOAs in the q-th Monte Carlo trial, and Q is the number of Monte Carlo trials. Here we set $Q = 500$. Firstly, the number of snapshots is fixed at 200, and the SNR varies from −10 dB to 10 dB, the results of which are shown in Fig. 4. Then, the SNR is fixed at −5 dB, and the number of snapshots varies from 50 to 500, the results of which are shown in Fig. 5.

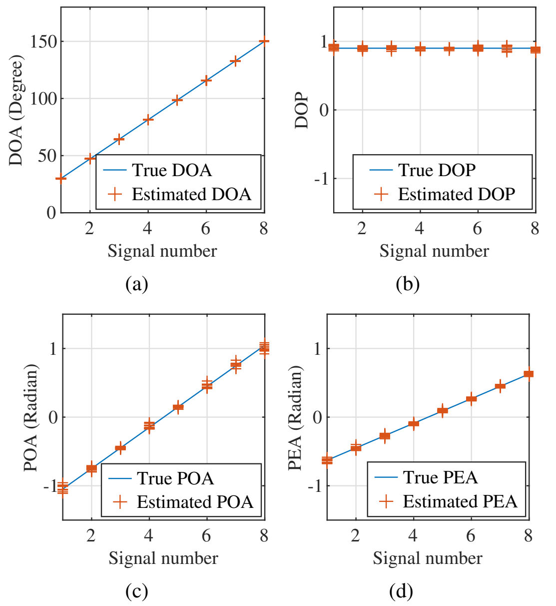

Fig. 3. The estimated DOAs and polarization parameters of ten trials: (a) DOA, (b) DOP, (c) POA, (d) PEA.

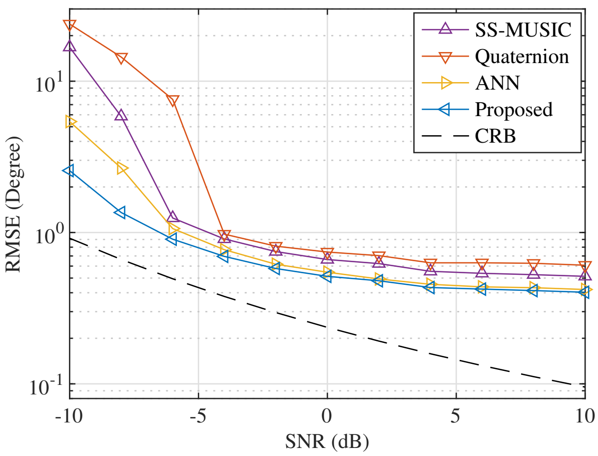

Fig. 4. RMSE of DOA estimation under different SNRs.

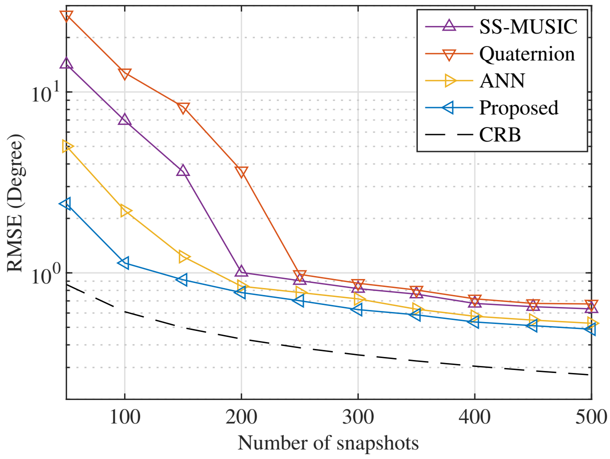

Fig. 5. RMSE of DOA estimation under different number of snapshots.

From Fig. 4 and Fig. 5, it can be found that the accuracy of the Quaternion method is the worst. This is because the Quaternion requires the powers of the horizontal and vertical polarization components of a signal to be equal. However, this requirement has not been satisfied here. The ANN is better than SS-MUSIC and Quaternion. The proposed method has the highest accuracy under different SNRs and under different numbers of snapshots.

In the third simulation, the proposed method and the other methods are compared with respect to the DOP estimation performance. Since the SS-MUSIC method can not estimate the DOP, it is omitted in this simulation. The performance is evaluated by the RMSE of DOP estimation, which is defined by

$$\mathrm{RMSE_{DOP}} = \sqrt{\frac{1}{Q}\sum_{q=1}^{Q}\frac{\|\boldsymbol{\rho}-\hat{\boldsymbol{\rho}}^{(q)}\|_2^2}{K}} \tag{33}$$

where $\boldsymbol{\rho} = [\rho_1, \rho_2, \ldots, \rho_K]$, and $\hat{\boldsymbol{\rho}}^{(q)}$ contains the estimated DOPs in the q-th Monte Carlo trial. Here we set $Q = 500$. Firstly, the number of snapshots is fixed at 500, and the SNR varies from –10 dB to 10 dB, the results of which are shown in Fig. 6. Then, the SNR is fixed at 5 dB, and the number of snapshots varies from 50 to 500, the results of which are shown in Fig. 7. It can be seen that the DOP estimation accuracy of the proposed method is far better than that of the Quaternion method. This performance difference comes from three aspects. Firstly, the proposed method adopts the tri-polarized configuration, which can completely accumulate the power of a signal. Secondly, the noise variance is estimated in the proposed method, which leads to an unbiased estimation of DOP. Thirdly, the proposed method does not require the powers of the horizontal and vertical polarization components of a signal to be equal. However, under low SNR situation, the proposed method has lower DOP estimation accuracy than the ANN method. This is because, under low SNR situation, the ANN can estimate the noise variance more accurately than the proposed method, leading to a more accurate reconstruction of the coherency matrix.

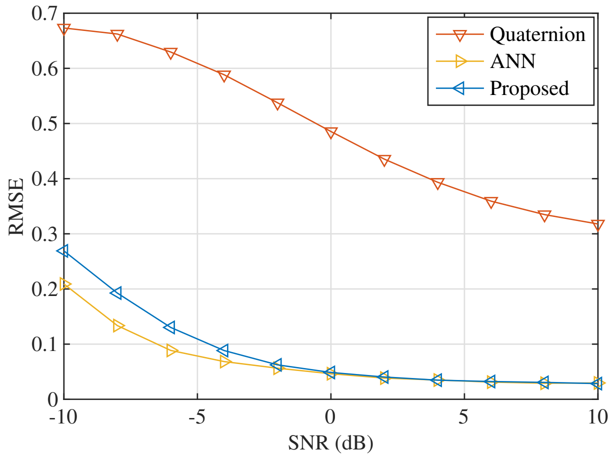

Fig. 6. RMSE of DOP estimation under different SNRs.

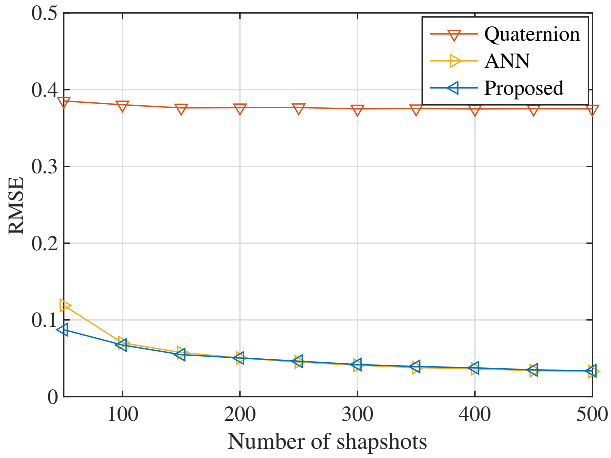

Fig. 7. RMSE of DOP estimation under different number of snapshots.

Algorithm	SNR = –10 dB	SNR = 0 dB	SNR = 10 dB
Quaternion	1.2e–3	1.1e–3	1.2e–3
ANN	9.2e–3	5.6e–3	5.4e–3
Proposed	6.0e–4	5.9e–4	6.1e–4

Tab. 2. The runtime comparison under different SNRs, unit: sec.

At last, the proposed method and the other methods are compared with respect to the computational complexity. The SS-MUSIC method is omitted in this simulation, since it does not estimate the DOP. The number of snapshots is fixed at 200, and the SNR is set as –10 dB, 0 dB, and 10 dB. The runtime of different methods under different situation is shown in Tab. 2. We can find the ANN is the slowest method since it needs iterative calculations. Also, it consumes more time under low SNR situation than under high SNR situation. This is because the number of iterations in ANN under low SNR situation is larger than that under high SNR situation. The proposed method has the highest speed, and it is about one order of magnitude faster than the ANN method.

5. Conclusion

In this paper, we introduce a non-iterative method for estimating the DOAs and polarizations of multiple PP signals using the tri-polarized nested array. First, with the sub-covariance addition, the power of different polarized components of a signal can be completely accumulated, which improves the SNR. Besides, it is proven that the noise variance estimation without iterations becomes possible in the underdetermined case, provided the tri-polarized nested array is designed with a given optimization rule. The new proposed method does not require iterative calculations. It is validated by numerical experiments and compared with other representative methods. The simulation results show that the proposed method possesses high efficiency and relatively high accuracy.

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Appendix A: Proof of Theorem 1

Since the signals are uncorrelated, $\mathbf{R}_s$ is a block diagonal matrix, and each submatrix on the diagnoal is called the coherency matrix, which has dimension 2×2. For a PP signal, its coherency matrix has rank 2. So, $\mathbf{R}_s$ has full rank, which means the covariance matrix $\mathbf{R}_s$ is positive definite. Thus, we can find an invertible matrix $\mathbf{Q}$ satisfying $\mathbf{R}_s = \mathbf{Q}\mathbf{Q}^{\mathrm{H}}$. Then we have

$$\begin{aligned}\mathrm{rank}(\mathbf{B}\mathbf{R}_s\mathbf{B}^{\mathrm{H}}) &=\mathrm{rank}(\mathbf{B}\mathbf{Q}\mathbf{Q}^{\mathrm{H}}\mathbf{B}^{\mathrm{H}})\\ &=\mathrm{rank}(\mathbf{B}\mathbf{Q})\\ &=\mathrm{rank}(\mathbf{B}).\end{aligned} \tag{A1}$$

Since $\mathbf{B}_{\cdot,k} = \mathbf{a}(\theta_k) \otimes \mathbf{C}_k$, and exchanging rows or columns does not change the matrix rank, we have

$$\begin{aligned}\mathrm{rank}(\mathbf{B}) &=\mathrm{rank}\left(\begin{bmatrix}-\mathbf{A} & \\ & \mathbf{D}\end{bmatrix}\right)\\ &=\mathrm{rank}(\mathbf{A}) + \mathrm{rank}(\mathbf{D})\end{aligned} \tag{A2}$$

where $\mathbf{A} \in \mathbb{C}^{L\times K} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_K)]$ and $\mathbf{D}_{\cdot,k} = [\sin\theta_k, -\cos\theta_k]^{\mathrm{T}} \otimes \mathbf{a}(\theta_k)$.

Because the nested array is a non-uniform array, $\mathbf{A}$ is a Vandermonde matrix with deleted rows. So, if $K < L$, $\mathrm{rank}(\mathbf{A}) \le K$, and if $K \ge L$, $\mathrm{rank}(\mathbf{A}) \le L$. Similarly, for $\mathbf{D}$, if $K < 2L$, $\mathrm{rank}(\mathbf{D}) \le K$, and if $K \ge 2L$, $\mathrm{rank}(\mathbf{D}) \le 2L$. So, combing the rank properties of $\mathbf{A}$ and $\mathbf{D}$, we have

$$\mathrm{rank}(\mathbf{B}) \begin{cases} \le 2K, & \text{if } K < L \\ \le L + K, & \text{if } L \le K < 2L \\ \le 3L, & \text{if } K \ge 2L \end{cases}, \tag{A3}$$

Therefor, according to (A1), $\mathbf{B}\mathbf{R}_s\mathbf{B}^{\mathrm{H}}$ has the rank property shown in (9).

Appendix B: Proof of (30)

Substituting (5) and (6) into (29), we have

$$\begin{aligned}&\xi_{k,1}\cos(\alpha)\cos(\beta) - \mathrm{j}\xi_{k,1}\sin(\alpha)\sin(\beta)\\ &- \xi_{k,2}\sin(\alpha)\cos(\beta) - \mathrm{j}\xi_{k,2}\cos(\alpha)\sin(\beta) = 0.\end{aligned} \tag{B1}$$

Then, substituting

$$\cos\alpha = \frac{\mathrm{e}^{\mathrm{j}\alpha} + \mathrm{e}^{-\mathrm{j}\alpha}}{2}, \quad \sin\alpha = \frac{\mathrm{e}^{\mathrm{j}\alpha} - \mathrm{e}^{-\mathrm{j}\alpha}}{2\mathrm{j}} \tag{B2}$$

into (B1) and using some mathematical simplification, we have

$$(\mathrm{j}\xi_{k,2}+\xi_{k,1})(1-\tan\beta)\mathrm{e}^{\mathrm{j}2\alpha} = (\mathrm{j}\xi_{k,2}-\xi_{k,1})(1+\tan\beta). \tag{B3}$$

If $\beta \ne -\pi/4$, dividing both sides of (B3) by $1 + \tan\beta$, we have

$$\tan(\pi/4 - \beta)\mathrm{e}^{\mathrm{j}2\alpha} = \frac{\mathrm{j}\xi_{k,2} - \xi_{k,1}}{\mathrm{j}\xi_{k,2} + \xi_{k,1}}. \tag{B4}$$

If $\beta = -\pi/4$, $\mathrm{j}\xi_{k,2} + \xi_{k,1} = 0$ can be obtained from (B3). According to (B4), $\mathrm{j}\xi_{k,2} + \xi_{k,1} = 0$ can also lead to $\beta = -\pi/4$ since $\beta \in [-\pi/4, \pi/4]$. Thus, (B4) holds for any β. Eventually (30) can be inferred from (B4).

Appendix C: CRB Derivation

The CRB of tri-polarization configuration for PP signals that is valid under overdetermined and underdetermined cases has not been analyzed in the literature. Here, we follow [17] to derive this new CRB. The Fisher information matrix (FIM) is expressed as

$$\mathrm{FIM} = N\left(\frac{\partial\mathbf{r}}{\partial\boldsymbol{\Omega}}\right)^{\mathrm{H}}(\mathbf{R}^{-\mathrm{T}} \otimes \mathbf{R}^{-1})\left(\frac{\partial\mathbf{r}}{\partial\boldsymbol{\Omega}}\right) \tag{C1}$$

where

$$\begin{aligned}\mathbf{r} &= \mathrm{vec}(\mathbf{R})\\ &= (\mathbf{B}^* \otimes \mathbf{B})\mathrm{vec}(\mathbf{R}_s) + \sigma_n^2\mathrm{vec}(\mathbf{I}_{3L})\\ &= \sum_{k=1}^{K}\mathbf{V}(\theta_k)\mathrm{vec}(\mathbf{R}_{s_k}) + \sigma_n^2\mathrm{vec}(\mathbf{I}_{3L})\end{aligned} \tag{C2}$$

and $\mathbf{V}(\theta_k) = [\mathbf{a}(\theta_k) \otimes \mathbf{C}_k]^* \otimes [\mathbf{a}(\theta_k) \otimes \mathbf{C}_k]$. $\boldsymbol{\Omega}$ containing all the unknown parameters is defined by

$$\boldsymbol{\Omega} = [\boldsymbol{\theta}^{\mathrm{T}}, \boldsymbol{\sigma}_u^{\mathrm{T}}, \boldsymbol{\sigma}_c^{\mathrm{T}}, \boldsymbol{\alpha}^{\mathrm{T}}, \boldsymbol{\beta}^{\mathrm{T}}, \sigma_n^2]^{\mathrm{T}}, \tag{C3}$$

where $\boldsymbol{\sigma}_u = [\sigma_{1,u}^2, \ldots, \sigma_{K,u}^2]^{\mathrm{T}}$, $\boldsymbol{\sigma}_c = [\sigma_{1,c}^2, \ldots, \sigma_{K,c}^2]^{\mathrm{T}}$, $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_K]^{\mathrm{T}}$, and $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_K]^{\mathrm{T}}$. Then, $\partial\mathbf{r}/\partial\boldsymbol{\Omega}$ can be written as

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\boldsymbol{\Omega}} = \Bigg[&\frac{\partial\mathbf{r}}{\partial\theta_1}, \ldots, \frac{\partial\mathbf{r}}{\partial\theta_K}, \frac{\partial\mathbf{r}}{\partial\sigma_{1,u}^2}, \ldots, \frac{\partial\mathbf{r}}{\partial\sigma_{K,u}^2}, \frac{\partial\mathbf{r}}{\partial\sigma_{1,c}^2}, \ldots,\\ &\frac{\partial\mathbf{r}}{\partial\sigma_{K,c}^2}, \frac{\partial\mathbf{r}}{\partial\alpha_1}, \ldots, \frac{\partial\mathbf{r}}{\partial\alpha_K}, \frac{\partial\mathbf{r}}{\partial\beta_1}, \ldots, \frac{\partial\mathbf{r}}{\partial\beta_K}, \frac{\partial\mathbf{r}}{\partial\sigma_n^2}\Bigg].\end{aligned} \tag{C4}$$

According to (C2), we have

$$\frac{\partial \mathbf{r}}{\partial \theta_k} = \left[\frac{\partial(\mathbf{a}(\theta_k) \otimes \mathbf{C}_k)^*}{\partial \theta_k} \otimes (\mathbf{a}(\theta_k) \otimes \mathbf{C}_k) + (\mathbf{a}(\theta_k) \otimes \mathbf{C}_k)^* \otimes \frac{\partial(\mathbf{a}(\theta_k) \otimes \mathbf{C}_k)}{\partial \theta_k} \right] \mathrm{vec}(\mathbf{R}_s), \tag{C5}$$

$$\frac{\partial \mathbf{r}}{\partial \sigma_{k,u}^2} = \mathbf{V}(\theta_k)[1/2, 0, 0, 1/2]^{\mathrm{T}}, \tag{C6}$$

$$\frac{\partial \mathbf{r}}{\partial \sigma_{k,c}^2} = \mathbf{V}(\theta_k)\mathrm{vec}\left(\mathbf{G}(\alpha_k)\mathbf{w}(\beta_k)\mathbf{w}^{\mathrm{H}}(\beta_k)\mathbf{G}^{\mathrm{H}}(\alpha_k)\right), \tag{C7}$$

$$\frac{\partial \mathbf{r}}{\partial \alpha_k} = \sigma_{k,c}^2 \mathbf{V}(\theta_k)(\left[\frac{\partial \mathbf{G}^*(\alpha_k)}{\partial \alpha_k} \otimes \mathbf{G}(\alpha_k) + \mathbf{G}^*(\alpha_k) \otimes \frac{\partial \mathbf{G}(\alpha_k)}{\partial \alpha_k} \right] \mathrm{vec}\left(\mathbf{w}(\beta_k)\mathbf{w}^{\mathrm{H}}(\beta_k)\right), \tag{C8}$$

$$\frac{\partial \mathbf{r}}{\partial \beta_k} = \sigma_{k,c}^2 \mathbf{V}(\theta_k)\, [\mathbf{G}^*(\alpha_k) \otimes \mathbf{G}(\alpha_k)]\, \mathrm{vec}\left(\frac{\partial \mathbf{w}(\beta_k)}{\partial \beta_k}\mathbf{w}^{\mathrm{H}} + \mathbf{w}\frac{\partial \mathbf{w}^{\mathrm{H}}(\beta_k)}{\partial \beta_k}\right), \tag{C9}$$

and

$$\frac{\partial \mathbf{r}}{\partial \sigma_n^2} = \mathrm{vec}(\mathbf{I}_{3M}) \tag{C10}$$

where $\partial(\mathbf{a}(\theta_k) \otimes \mathbf{C}_k)/\partial\theta_k$, $\partial\mathbf{G}(\alpha_k)/\partial\alpha_k$ and $\partial\mathbf{w}(\beta_k)/\partial\beta_k$ can be easily derived, respectively. Then, substituting (C4)–(C10) into (C1), we can achieve the FIM. Therefore, the CRB of the DOA estimation for the k-th PP signal can be obtained by

$$\mathrm{CRB}(\theta_k) = [\mathbf{FIM}^{-1}]_{k,k}, \quad k = 1, 2, \ldots, K. \tag{C11}$$