Unified Expression of the Conjugate Image Impedances for Two-port Representations

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Abstract. *Conjugate image impedances are used to minimize power reflections in a variety of domains, including amplifier design, microwave engineering, wireless power transfer, antenna design and millimeter wave applications. For a two-port network, they can be described as function of different parameters including impedance, admittance, hybrid, inverse hybrid, chain, scattering and chain scattering parameters. In this work, a general unified structure for the conjugate image impedances is provided, valid for each of the two-port representations. It highlights its close relationship with the Rollett stability factor and provides insight into the structure of conjugate image impedances.*

Keywords

Circuit theory, conjugate image, two-port networks

1. Introduction

Conjugate image impedances are used to minimize power reflections in a variety of domains, including amplifier design [1], microwave engineering [2], stub tuning [2], wireless power transfer [3], [4], filter design [5], transmission lines [6], antenna design [7], reciprocal power transfer systems [8] and millimeter wave applications [9].

They can be calculated as function of the elements of the impedance matrix $\mathbf{Z}$ or the dual admittance matrix $\mathbf{Y}$ of a two-port network. For systems that are described as function of other parameters such as hybrid (h), inverse hybrid (g), chain (a, b), scattering (S) and chain scattering (T) parameters, one can first convert the two-port matrix to its impedance representation (e.g., [10], [11]), and then calculate the conjugate image impedance.

The purpose of the present work is to provide a unified structure for the conjugate image impedances for the different matrix representations, and highlight its close relationship with the Rollett stability factor. The scope of the work is

mainly academic: in a practical setting, the well-known immittance parameter conversions (e.g., [10], [11]) can be easily used to attain the conjugate images for each representation. Notwithstanding our modest ambitions, the motivation of the present paper is to provide further insight into the structure of conjugate image impedances, which may offer new ways to understand how conjugate image impedances could be provided for multiport networks. Indeed, determination of the conjugate image impedances has only been feasible so far for two-port networks only [12]. Moreover, our unified structure also includes the representation in S and T parameters with complex normalization impedances.

We will first recall the conjugate image impedances on the bases of an example of wireless power transfer (Sec. 2). Next, we express the Rollett stability factor as function of the different network parameters in Sec. 3, which will lead to a unified expression for the conjugate image immittances in Sec. 4. Finally, we illustrate the unified equation by some examples (Sec. 5).

2. Preliminaries

Consider as numerical example the equivalent circuit of an inductive wireless power transfer system (Fig. 1). The circuit can be considered as a two-port network with a source V_S connected to port #1 and a load Z_L connected to port #2. In this example, power is transferred via coupled coils with coupling factor $k_L = 10\%$ from the source V_S (with frequency $f = 100$ kHz) to the load Z_L. For the values indicated in the figure, it can be calculated that the power conversion coefficient is 23.4% when Z_L equals $50\,\Omega$.
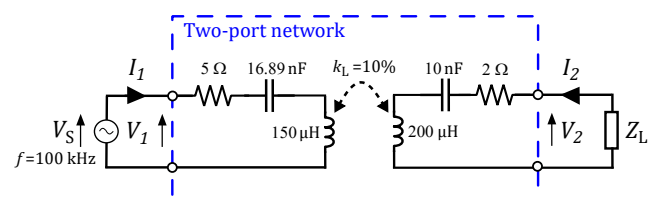

Fig. 1. Equivalent circuit of an inductive wireless power transfer system as example of a two-port.

It is well known that, in order to realize a higher power transfer efficiency, impedance matching can be applied to minimize power reflections. We name Z_{c1} and Z_{c2} the impedances that realize minimum reflections at port #1 and #2, respectively. These impedances are called the conjugate image impedances.

For example, it can be calculated [13] that the conjugate image impedance Z_{c2} of port #2 is given by $Z_{c2} = 7.17\,\Omega + j33.49\,\Omega$. Applying this load to the network of Fig. 1 realizes impedance matching and results in the maximum available power conversion coefficient of the given two-port, which can be calculated to be 56.4%. Indeed, the impedance matching reduces the reflected power and maximizes the power transfer from one port to another.

The conjugate image impedances for a two-port network are defined as (Fig. 2) [1]:

- If we terminate port #2 with a load Z_{c2}, the input impedance as seen into port #1 is Z_{c1}^*

- If we terminate port #1 with an impedance Z_{c1}, the impedance as seen into port #2 is Z_{c2}^*

The superscript * indicates the complex conjugate.

The conjugate image impedances are intrinsic properties of the two-port network and are only dependent on the two-port network itself. They are independent on the connected impedances to the ports. The value of Z_{c1} and Z_{c2} as function of the elements $z_{ij} = r_{ij} + jx_{ij}$ $(i, j = 1, 2)$ of the impedance matrix $\mathbf{Z}$ of the two-port network can be easily determined by expressing the conditions for Fig. 2 into a system of equations. The input impedance Z_{in} at port #1 when Z_{c2} is connected to port #2 is:

$$Z_{in} = z_{11} - \frac{z_{12}z_{21}}{z_{22}Z_{c2}}. \tag{1}$$

The impedance Z_{out} at port #2 when Z_{c1} is connected to port #1 is:

$$Z_{out} = z_{22} - \frac{z_{12}z_{21}}{z_{11}Z_{c1}}. \tag{2}$$

Solving the above system in Z_{c1} and Z_{c2} when $Z_{in} = Z_{c1}^*$ and $Z_{out} = Z_{c2}^*$ results in the solution for the conjugate image impedances as function of the elements of the impedance matrix $\mathbf{Z}$:

$$Z_{c1} = r_{11}(\theta_r + j\theta_x) - jx_{11}, \tag{3}$$

$$Z_{c2} = r_{22}(\theta_r + j\theta_x) - jx_{22} \tag{4}$$

with auxiliary variables:

$$\theta_x = \frac{r_{12}x_{21} + r_{21}x_{12}}{2r_{11}r_{22}}, \tag{5}$$

$$\theta_r = \sqrt{1 - \theta_x^2 + \frac{x_{12}x_{21} - r_{12}r_{21}}{r_{11}r_{22}}}. \tag{6}$$
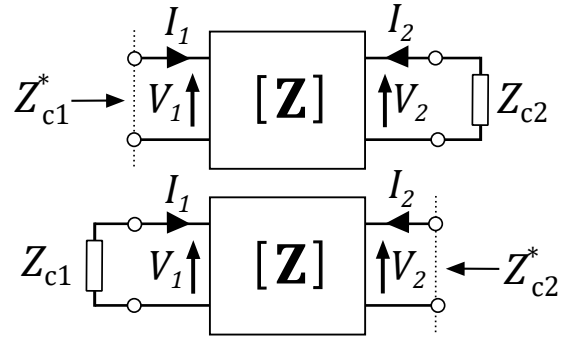

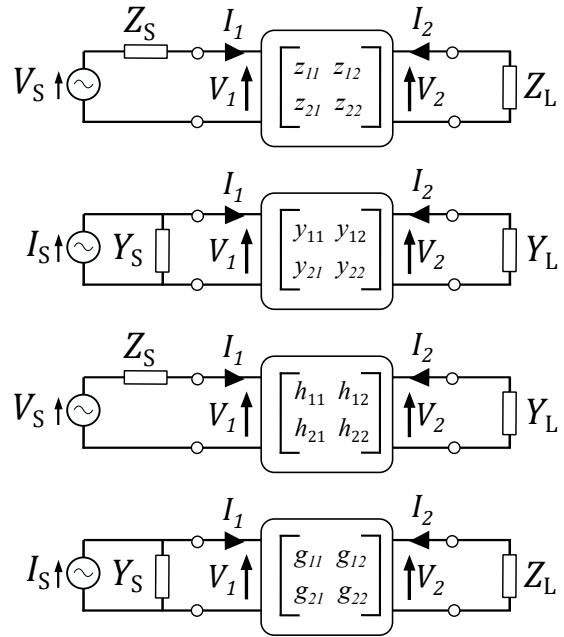**Fig. 2.** Definition of the conjugate image impedances Z_{c1} and Z_{c2}.

Fig. 3. Immittance representations of a two-port network: impedance (z), admittance (y), hybrid (h) and inverse hybrid (g) parameters.

For the wireless power transfer example of Fig. 1, the impedance matrix $\mathbf{Z}$ can be calculated as:

$$\mathbf{Z} = \begin{bmatrix} 5\,\Omega & j10.88\,\Omega \\ j10.88\,\Omega & 2\,\Omega - j33.49\,\Omega \end{bmatrix}. \tag{7}$$

Applying (3) and (4) results in the conjugate image impedances $Z_{c1} = 17.91\,\Omega$ and $Z_{c2} = 7.17\,\Omega + j33.49\,\Omega$ that realize impedance matching at their respective ports. Obviously, we can also express these values as conjugate image admittances $Y_{c1} = 1/Z_{c1} = 55.8\,mS$ and $Y_{c2} = 1/Z_{c2} = 6.1\,mS - j28.6\,mS$. In the remainder of this work, we will apply the notation Γ_{c1} and Γ_{c2} for the conjugate image immittances, which are either impedances or admittances, depending on the context.

Each network representation has some specific advantages over the others, depending on the application. For example, consider the four immittance representations of Fig. 3.

If a voltage source is present at the input port, either the impedance parameters z or the hybrid parameters h are convenient. If besides the voltage source, a parallel load is present, the hybrid parameters h can be best suited. In this work, all four immittance parameters will be indicated by the symbol

$$\gamma_{ij} = \rho_{ij} + \mathrm{j}\sigma_{ij}. \tag{8}$$

For example, in the case of impedance parameters, we obtain $\gamma_{ij} = z_{ij} = r_{ij} + \mathrm{j}x_{ij}$. Depending on the representation, the immittance γ_{ij} has the unit of Ω or S.

In appendix A, we recall the definitions of the different two-port matrix representations, in particular because different definitions can be found in literature, in particular with regard to S parameters with complex normalization impedances.

3. Rollett Stability Factor

Before continuing, it is useful to recall the dimensionless Rollett stability factor k, which is for the immittance parameters defined as [14], [15]:

$$k = \frac{2\rho_{11}\rho_{22} - \Re(\gamma_{12}\gamma_{21})}{|\gamma_{12}\gamma_{21}|}. \tag{9}$$

The Rollett stability factor is applied in the study of amplifier circuits. It characterizes the stability condition of transistors, and its input and output adaptation, attenuation, losses and phase in an amplifier circuit.

It is invariant for the four different immittances. In other words, the same value of k is obtained, no matter which immittance representation is chosen. E.g., for a given two-port network with impedance elements $z_{ij} = r_{ij} + \mathrm{j}x_{ij}$ and corresponding admittance elements $y_{ij} = g_{ij} + \mathrm{j}b_{ij}$, one can write:

$$k = \frac{2r_{11}r_{22} - \Re(z_{12}z_{21})}{|z_{12}z_{21}|} = \frac{2g_{11}g_{22} - \Re(y_{12}y_{21})}{|y_{12}y_{21}|}. \tag{10}$$

Since the Rollett stability factor k will appear in the expression for the conjugate image immittances, it is useful to express its value as a function of the other representations. As function of the chain parameters a and b, its expression can be determined from [15], [16]. We find:

$$k = \frac{\Re(a_{12}a_{21}^*) + \Re(a_{11}^*a_{22})}{|\Delta\mathbf{a}|} = \frac{\Re(b_{12}b_{21}^*) + \Re(b_{11}^*b_{22})}{|\Delta\mathbf{b}|} \tag{11}$$

with $|\Delta\mathbf{a}|$ the modulus of the determinant of the matrix $\mathbf{a}$, consisting of elements $a_{ij}(i, j = 1, 2)$. Notice that the Rollett stability factor is invariant for the a and b parameters.

	k
Immittance	$\dfrac{2\rho_{11}\rho_{22} - \Re(\gamma_{12}\gamma_{21})}{|\gamma_{12}\gamma_{21}|}$
a parameters	$\dfrac{\Re(a_{12}a_{21}^*) + \Re(a_{11}^*a_{22})}{|\Delta\mathbf{a}|}$
b parameters	$\dfrac{\Re(b_{12}b_{21}^*) + \Re(b_{11}^*b_{22})}{|\Delta\mathbf{b}|}$
S parameters	$\dfrac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta\mathbf{S}|^2}{2|S_{12}S_{21}|}$
T parameters	$\dfrac{|T_{11}|^2 + |T_{22}|^2 - |T_{12}|^2 - |T_{21}|^2}{2|\Delta\mathbf{T}|}$

Tab. 1. The Rollett stability factor k as function of the two-port parameters. For the S and T parameters, the normalization impedances equal $Z_{01} = Z_{02} = 1\,\Omega$.

The Rollett stability factor k for the S and T parameters can also be derived from the series of steps described in [15], [16]. We obtain:

$$k = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta\mathbf{S}|^2}{2|S_{12}S_{21}|}, \tag{12}$$

$$k = \frac{|T_{11}|^2 + |T_{22}|^2 - |T_{12}|^2 - |T_{21}|^2}{2|\Delta\mathbf{T}|}, \tag{13}$$

for normalization impedances $Z_{01} = Z_{02} = 1\,\Omega$, valid for both definitions of S and T parameters. We refer to [6, 17, 18] for converting to other (complex) normalization impedances.

We emphasize that the Rollett stability factor k is inherently a characteristic of the two-port network, and has the same value, no matter the chosen representation matrix. This also implies that the value of k is independent on the chosen normalization impedances of the S or T parameters. Table 1 gives an overview of k expressed as function of the different two-port representations.

4. Conjugate Immittances

In this section, we first express the conjugate image immittances as function of the different two-port representations. This will allow for a unified expression.

4.1 As Function of the Immittance Parameters

The conjugate immittances Γ_{c1} and Γ_{c2} which realize conjugate match at the ports can be expressed in terms of the Rollett stability factor [14], [19]:

$$\Gamma_{c1} = \frac{\gamma_{12}\gamma_{21} + |\gamma_{12}\gamma_{21}|(k + \sqrt{k^2 - 1})}{2\rho_{22}} - \gamma_{11}, \tag{14}$$

$$\Gamma_{c2} = \frac{\gamma_{12}\gamma_{21} + |\gamma_{12}\gamma_{21}|(k + \sqrt{k^2 - 1})}{2\rho_{11}} - \gamma_{22}. \tag{15}$$

It can be shown that they correspond to the values of (3) and (4) in the impedance representation.

4.2 As Function of the Chain Parameters

By expressing (14) and (15) as function of the impedance parameters z, and applying the parameter conversion [10], [11]:

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \frac{1}{a_{21}} \begin{bmatrix} a_{11} & \Delta\mathbf{a} \\ 1 & a_{22} \end{bmatrix} \qquad (16)$$

we obtain:

$$\Gamma_{c1} = \frac{\gamma_{a12} + |\gamma_{a12}|(k + \sqrt{k^2 - 1})}{2\Re(\gamma_{a22})} - \gamma_{a11}, \qquad (17)$$

$$\Gamma_{c2} = \frac{\gamma_{a12} + |\gamma_{a12}|(k + \sqrt{k^2 - 1})}{2\Re(\gamma_{a11})} - \gamma_{a22} \qquad (18)$$

with

$$\gamma_{a11} = \frac{a_{11}}{a_{21}}, \qquad (19)$$

$$\gamma_{a22} = \frac{a_{22}}{a_{21}}, \qquad (20)$$

$$\gamma_{a12} = \frac{\Delta\mathbf{a}}{a_{21}^2}. \qquad (21)$$

This results in expressions for the conjugate image immittances Γ_{c1} and Γ_{c2}, written entirely in a parameters. In other words, one can directly determine Γ_{c1} and Γ_{c2} from the a parameter representation, without first converting to, e.g. an impedance representation z. Notice that also k can be calculated from the a parameters as indicated in Tab. 1.

Analogous, by expressing (14) and (15) as function of the impedance parameters z, and applying the parameter conversion [10], [11]:

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = -\frac{1}{b_{21}} \begin{bmatrix} b_{22} & 1 \\ \Delta\mathbf{b} & b_{11} \end{bmatrix} \qquad (22)$$

we obtain:

$$\Gamma_{c1} = \frac{\gamma_{b12} + |\gamma_{b12}|(k + \sqrt{k^2 - 1})}{2\Re(\gamma_{b22})} - \gamma_{b11}, \qquad (23)$$

$$\Gamma_{c2} = \frac{\gamma_{b12} + |\gamma_{b12}|(k + \sqrt{k^2 - 1})}{2\Re(\gamma_{b11})} - \gamma_{b22} \qquad (24)$$

with

$$\gamma_{b11} = -\frac{b_{22}}{b_{21}}, \qquad (25)$$

$$\gamma_{b22} = -\frac{b_{11}}{b_{21}}, \qquad (26)$$

$$\gamma_{b12} = \frac{\Delta\mathbf{b}}{b_{21}^2}. \qquad (27)$$

4.3 As Function of the S and T Parameters

The impedance parameters z expressed as function of the S parameters with complex reference impedances Z_{01} and Z_{02} for port #1 and #2, respectively, are given by (see Appendix B):

$$z_{11} = \frac{Z_{01}}{n_S}(1 + S_{11} - S_{22} - \Delta\mathbf{S}), \qquad (28)$$

$$z_{12} = 2\frac{Z_{02}}{n_S}\frac{p_2}{p_1}S_{12}, \qquad (29)$$

$$z_{21} = 2\frac{Z_{01}}{n_S}\frac{p_1}{p_2}S_{21}, \qquad (30)$$

$$z_{22} = \frac{Z_{02}}{n_S}(1 - S_{11} + S_{22} - \Delta\mathbf{S}) \qquad (31)$$

with $\Delta\mathbf{S}$ the determinant of $\mathbf{S}$ and

$$n_S = 1 - S_{11} - S_{22} + \Delta\mathbf{S}. \qquad (32)$$

These expressions are substituted into (14) and (15) in order to obtain the conjugate image immitances as function of the S parameters:

$$\Gamma_{c1} = \frac{\gamma_{S12} + |\gamma_{S12}|(k + \sqrt{k^2 - 1})}{2\Re(\gamma_{S22})} - \gamma_{S11}, \qquad (33)$$

$$\Gamma_{c2} = \frac{\gamma_{S12} + |\gamma_{S12}|(k + \sqrt{k^2 - 1})}{2\Re(\gamma_{S11})} - \gamma_{S22} \qquad (34)$$

with

$$\gamma_{S11} = \frac{Z_{01}}{n_S}(1 + S_{11} - S_{22} - \Delta\mathbf{S}), \qquad (35)$$

$$\gamma_{S22} = \frac{Z_{02}}{n_S}(1 - S_{11} + S_{22} - \Delta\mathbf{S}), \qquad (36)$$

$$\gamma_{S12} = 4\frac{Z_{01}Z_{02}}{n_S^2}S_{12}S_{21}. \qquad (37)$$

Analogous, the impedance parameters z can be expressed as function of the T parameters with complex reference impedances Z_{01} and Z_{02} (see Appendix B):

$$z_{11} = \frac{Z_{01}}{n_T}(T_{11} + T_{12} + T_{21} + T_{22}), \qquad (38)$$

$$z_{12} = 2\frac{Z_{02}}{n_T}\frac{p_2}{p_1}\Delta\mathbf{T}, \qquad (39)$$

$$z_{21} = 2\frac{Z_{01}}{n_T}\frac{p_1}{p_2}, \qquad (40)$$

$$z_{22} = \frac{Z_{02}}{n_T}(T_{11} - T_{12} - T_{21} + T_{22}) \qquad (41)$$

with $\Delta\mathbf{T}$ the determinant of $\mathbf{T}$ and

$$n_{\mathrm{T}} = -T_{11} - T_{12} + T_{21} + T_{22}. \tag{42}$$

The conjugate image immittances as function of the T parameters are found by substitution of the above equations into (14) and (15):

$$\Gamma_{\mathrm{c1}} = \frac{\gamma_{T12} + |\gamma_{T12}|(k + \sqrt{k^2 - 1})}{2\mathfrak{R}(\gamma_{T22})} - \gamma_{T11}, \tag{43}$$

$$\Gamma_{\mathrm{c2}} = \frac{\gamma_{T12} + |\gamma_{T12}|(k + \sqrt{k^2 - 1})}{2\mathfrak{R}(\gamma_{T11})} - \gamma_{T22} \tag{44}$$

with

$$\gamma_{T11} = \frac{Z_{01}}{n_{\mathrm{T}}}(T_{11} + T_{12} + T_{21} + T_{22}), \tag{45}$$

$$\gamma_{T22} = \frac{Z_{02}}{n_{\mathrm{T}}}(T_{11} - T_{12} - T_{21} + T_{22}), \tag{46}$$

$$\gamma_{T12} = 4\frac{Z_{01}Z_{02}}{n_{\mathrm{T}}^2}\Delta\mathbf{T}. \tag{47}$$

4.4 Unified Expression

In the previous sections, the expression for the conjugate image immittances were derived for the different two port representations. These expressions can be summarized into the following unified equations, valid for all the given two-port representations:

$$\boxed{\begin{aligned} \Gamma_{\mathrm{c1}} &= \frac{\psi_{12} + |\psi_{12}|(k + \sqrt{k^2 - 1})}{2\mathfrak{R}(\psi_{22})} - \psi_{11} \end{aligned}} \tag{48}$$

$$\boxed{\begin{aligned} \Gamma_{\mathrm{c2}} &= \frac{\psi_{12} + |\psi_{12}|(k + \sqrt{k^2 - 1})}{2\mathfrak{R}(\psi_{11})} - \psi_{22} \end{aligned}} \tag{49}$$

The parameters ψ_{ij} are given in Tab. 2 for each representation. As already mentioned, Tab. 1 gives an overview of the Rollett stability factor k expressed as function of the different two-port representations.

Equations (48) and (49) provide a general structure for the conjugate image immittances of two-port networks, valid for the immittance representations z, y, h, g, the chain representations a and b, and the S and T parameters with different complex normalization impedances for each port. They are dependent on the Rollett stability factor k. As far as we can tell, this general structure has not yet been presented.

Since the Rollett stability factor k is always real, the first term of Γ_{c1} and Γ_{c2} is always real for reciprocal systems. As a result, the imaginary part of the conjugate image immittance $\Gamma_{\mathrm{c}i}$ for reciprocal systems equals ψ_{ii}, which corresponds to the self immittance of the corresponding port.

It is worthwhile noting that the unified approach introduces additional divisions due to the transformation into auxiliary parameters compared to e.g. (3) and (4). For calculations requiring very high precision or extreme outlier values, attention should be paid to the applied computational precision.

5. Numerical Examples

Consider the numerical wireless power transfer example from the introduction (Fig. 1). We calculate the conjugate image immittances for different representations from the unified expression.

5.1 Impedance Representation: z Parameters

The impedance matrix $\mathbf{Z}$ is given by (7). From (10), the Rollett stability factor follows:

$$k = \frac{20 - \mathfrak{R}[(\mathrm{j}10.88)^2]}{|(\mathrm{j}10.88)^2|} = 1.17. \tag{50}$$

From Tab. 2, we find

$$\psi_{11,z} = z_{11} = 5.0\,\Omega, \tag{51}$$

$$\psi_{22,z} = z_{22} = 2.0\,\Omega - \mathrm{j}33.49\,\Omega, \tag{52}$$

$$\psi_{12,z} = z_{12}z_{21} = -118.37\,\Omega^2. \tag{53}$$

	ψ_{11}	ψ_{22}	ψ_{12}
Immittance	γ_{11}	γ_{22}	$\gamma_{12}\gamma_{21}$
a parameters	$\frac{a_{11}}{a_{21}}$	$\frac{a_{22}}{a_{21}}$	$\frac{\Delta\mathbf{a}}{a_{21}^2}$
b parameters	$-\frac{b_{22}}{b_{21}}$	$-\frac{b_{11}}{b_{21}}$	$\frac{\Delta\mathbf{b}}{b_{21}^2}$
S parameters	$\frac{Z_{01}}{n_{\mathrm{S}}}(1 + S_{11} - S_{22} - \Delta\mathbf{S})$	$\frac{Z_{02}}{n_{\mathrm{S}}}(1 - S_{11} + S_{22} - \Delta\mathbf{S})$	$4\frac{Z_{01}Z_{02}}{n_{\mathrm{S}}^2}S_{12}S_{21}$
T parameters	$\frac{Z_{01}}{n_{\mathrm{T}}}(T_{11} + T_{12} + T_{21} + T_{22})$	$\frac{Z_{02}}{n_{\mathrm{T}}}(T_{11} - T_{12} - T_{21} + T_{22})$	$4\frac{Z_{01}Z_{02}}{n_{\mathrm{T}}^2}\Delta\mathbf{T}$

with $n_{\mathrm{S}} = 1 - S_{11} - S_{22} + \Delta\mathbf{S}$ and $n_{\mathrm{T}} = -T_{11} - T_{12} + T_{21} + T_{22}$

Tab. 2. The parameters ψ_{ij} for different two-port representations.

Substitution of the numerical values into (48) and (49) results into the same values as found in the introduction:

$$\Gamma_{c1,z} = \frac{(j10.88)^2 + |(j10.88)^2|(k + \sqrt{k^2 - 1})}{4} - 5$$
$$= 17.91\,\Omega, \quad (54)$$

$$\Gamma_{c2,z} = \frac{(j10.88)^2 + |(j10.88)^2|(k + \sqrt{k^2 - 1})}{10} - 2 + j33.49$$
$$= 7.17\,\Omega + j33.49\,\Omega. \quad (55)$$

5.2 Hybrid Representation: *h* Parameters

If we suppose that the above wireless power transfer example was given via an *h* parameter representation, the two-port matrix would be given by:

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 5.21\,\Omega + j3.52\,\Omega & -0.32 + j0.019 \\ 0.32 - j0.019 & 0.0018\,\text{S} + j0.030\,\text{S} \end{bmatrix}.$$
$$(56)$$

The Rollett stability factor, calculated from the above *h* parameters, can be found in Tab. 1 and equals 1.17, the same value as the one calculated via the impedance parameters. From Tab. 2, we find

$$\psi_{11,h} = h_{11} = 5.21\,\Omega + j3.52\,\Omega, \quad (57)$$

$$\psi_{22,h} = h_{22} = 0.0018\,\text{S} + j0.030\,\text{S}, \quad (58)$$

$$\psi_{12,h} = h_{12}h_{21} = -0.10 + j0.013. \quad (59)$$

Substituting these numerical values into the unified expressions for the conjugate image immittances (48) and (49) result into an impedance at port #1 and an admittance at port #2:

$$\Gamma_{c1,h} = 17.91\,\Omega, \quad (60)$$

$$\Gamma_{c2,h} = 6.1\,\text{mS} - j29\,\text{mS}. \quad (61)$$

Obviously, $\Gamma_{c1,h} = \Gamma_{c1,z}$ and $\Gamma_{c2,h} = 1/\Gamma_{c2,z}$.

5.3 Chain Representation: *a* Parameters

If the wireless power transfer example would be represented by the *a* parameters, the two-port matrix would be given by:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -j0.46 & -15.39\,\Omega - j11.80\,\Omega \\ -j0.092\,\text{S} & -3.08 - j0.18 \end{bmatrix}. \quad (62)$$

The Rollett stability factor, calculated from the above *a* parameters, is given by (11) and again equals 1.17. The auxiliary parameters ψ_{ij} can also be easily determined from Tab. 2 and equal the values found for the impedance representation. Substitution into (48) and (49) results in the same conjugate impedances.

5.4 *S* Parameters

Consider the *S* matrix of the wireless power transfer configuration with e.g., normalization impedances $Z_{01} = 50\,\Omega$ for port #1 and $Z_{02} = 5\,\Omega - j20\,\Omega$ for port #2:

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} -0.81 + j0.071 & -0.47 + j0.080 \\ -0.0087 + j0.11 & 0.24 - j0.057 \end{bmatrix}. \quad (63)$$

From Tab. 2, we find, with $n_S = 1.38 + j0.104$, the same auxiliary parameters ψ_{ij} (and thus also conjugate image impedances) as for the impedance representation.

6. Conclusion

In order to minimize power reflections, conjugate image immittances Γ_{c1} and Γ_{c2} can be connected to the ports of a two-port network. However, a network can be characterized by different representations. In this work, we expressed the conjugate image immittances Γ_{c1} and Γ_{c2} as a unified equation, given by (48) and (49), and valid for the immittance representations z, y, h, g, the chain representations a and b, and the S and T parameters with different complex normalization impedances for each port. The Rollett stability factor k and auxiliary parameters ψ_{ij} for each representation are listed in Tabs. 1 and 2. In this way, we have provided a general structure for the conjugate image immittances of two-port networks.

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Appendix A: Definitions of Two-port Representations

In this appendix, we recall the different two-port matrix representations, in particular because different definitions can be found in literature. If other definitions are used, it is straightforward to adjust our results.

Immittance parameters: Taken into account the voltages and currents as defined in Fig. 3, the four immittance parameters are defined as:

- Impedance parameters z

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{A1}$$

- Admittance parameters y

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{A2}$$

- Hybrid parameters h

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}. \tag{A3}$$

- Inverse hybrid parameters g

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}. \tag{A4}$$

Chain parameters: We will apply the following definitions for the chain parameters (sometimes also referred to as transmission or ABCD parameters). These parameters simplify a.o. the cascading of two-ports.

- a parameters

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{A5}$$

- b parameters

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}. \tag{A6}$$

S and T parameters: Whereas the immittance and chain parameters are only dependent on the characteristics of the two-port itself, the S (scattering) and T (chain scattering or scattering transfer) parameters are dependent on chosen normalization impedances Z_{01} and Z_{02}. Common practical values are equal normalization impedances (resistances) of $1\,\Omega$, $50\,\Omega$ or $75\,\Omega$. However, different normalization impedances can be chosen for each port. Moreover, these normalization impedances can be complex values (with positive real parts) [6], [10].

Two distinct definitions are found in literature for the S and T parameters with complex reference impedances. They are incompatible with each other, and each serve a different purpose, depending on the application. We refer to [6, 20–23] for an in depth overview.

The first definition relates the S and T parameters to the normalized wave amplitudes of the voltage waves a_i and b_i at each port ($i = 1, 2$):

$$a_i = p_i(V_i + Z_{0i}I_i), \tag{A7}$$

$$b_i = p_i(V_i - Z_{0i}I_i) \tag{A8}$$

with

$$p_i = \frac{\sqrt{\Re(Z_{0i})}}{2|Z_{0i}|}. \tag{A9}$$

The other definition relates to the power waves $a_{p,i}$ and $b_{p,i}$ by ($i = 1, 2$):

$$a_{p,i} = p_{p,i}(V_i + Z_{0i}I_i), \tag{A10}$$

$$b_{p,i} = p_{p,i}(V_i - Z_{0i}^*I_i) \tag{A11}$$

with

$$p_{p,i} = \frac{1}{2\sqrt{\Re(Z_{0i})}}. \tag{A12}$$

In this work, equations (A7) and (A8) are used for the S and T parameters, since this definition is essential for certain applications, which is not the case for (A10) and (A11) [21].

The S and T matrices express the relations between the wave-based parameters:

- S parameters

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{A13}$$

- T parameters

$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}. \tag{A14}$$

Note that sometimes the T parameters are defined as $a_1 = T_{11}b_2 + T_{12}a_2$ and $b_1 = T_{21}b_2 + T_{22}a_2$. In this case, adaptation of our results is straightforward: one can simply interchange T_{11} for T_{22}, and T_{11} for T_{22}.

Appendix B: S and T Parameter Transformation

Transformation from one representation to another can easily be performed by matrix calculations. We recall the procedure set-up in [11], and apply it to transform S and T parameters to z parameters. We explicitly perform the derivation, since many sources (e.g., [11], [20]) assume the same reference impedance for each port, whereas we generalize to different reference impedances for each port.

In general, a representation $\mathbf{R}$ converts the input $\mathbf{U}$ to the output $\mathbf{O}$

$$\mathbf{O} = \mathbf{RU}, \tag{B1}$$

e.g., equation (A13) for S parameters. We wish to perform a transformation to another representation $\boldsymbol{R}^N$ with input $\mathbf{U}^N$ and output $\mathbf{O}^N$:

$$\mathbf{O}^N = \mathbf{R}^N\mathbf{U}^N, \tag{B2}$$

e.g., equation (A1) for z parameters. This transformation can be described by an (at this point unknown) transformation matrix $\mathbf{P}$ connecting the stacked input and output matrices

$$\begin{bmatrix} \mathbf{O}^N \\ \mathbf{U}^N \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{O} \\ \mathbf{U} \end{bmatrix} \tag{B3}$$

where we have divided the matrix $\mathbf{P}$ into submatrices $\mathbf{P}_{ij}$. Solving for $\mathbf{O}^N$ results into:

$$\mathbf{O}^N = (\mathbf{P}_{11}\mathbf{R} + \mathbf{P}_{12})(\mathbf{P}_{21}\mathbf{R} + \mathbf{P}_{22})^{-1}\mathbf{U}^N \tag{B4}$$

and thus from (B2)

$$\mathbf{R}^N = (\mathbf{P}_{11}\mathbf{R} + \mathbf{P}_{12})(\mathbf{P}_{21}\mathbf{R} + \mathbf{P}_{22})^{-1}. \tag{B5}$$

If we know the transformation matrix $\mathbf{P}$, this expression allows us to transform any representation to another. Expression (B3) can be written as

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{O}^N \\ \mathbf{U}^N \end{bmatrix} = \begin{bmatrix} \mathbf{O} \\ \mathbf{U} \end{bmatrix} \tag{B6}$$

and equals for the S parameters, based on (A7) and (A8):

$$\begin{bmatrix} p_1 & 0 & -p_1Z_{01} & 0 \\ 0 & p_2 & 0 & -p_2Z_{02} \\ p_1 & 0 & p_1Z_{01} & 0 \\ 0 & p_2 & 0 & p_2Z_{02} \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ a_1 \\ a_2 \end{bmatrix}. \tag{B7}$$

Inverting the first matrix results into the transformation matrix $\mathbf{P}_{S\rightarrow z}$ for the conversion from S to z parameters:

$$\mathbf{P}_{S\rightarrow z} = \frac{1}{2}\begin{bmatrix} \frac{1}{p_1} & 0 & \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} & 0 & \frac{1}{p_2} \\ -\frac{1}{p_1Z_{01}} & 0 & \frac{1}{p_1Z_{01}} & 0 \\ 0 & -\frac{1}{p_2Z_{02}} & 0 & -\frac{1}{p_2Z_{02}} \end{bmatrix}. \tag{B8}$$

Applying this transformation matrix $\mathbf{P}_{S\rightarrow z}$ into (B5) results into (28), (29), (30) and (31) that express the z parameters as function of the S parameters for different complex normalization impedances.

Note that the derivation for S to z parameters reduces to the results from [11], [20] for equal reference impedances.

Analogous, we obtain for the T parameters

$$\mathbf{P}_{T\rightarrow z} = \frac{1}{2}\begin{bmatrix} \frac{1}{p_1} & \frac{1}{p_1} & 0 & 0 \\ 0 & 0 & \frac{1}{p_2} & \frac{1}{p_2} \\ -\frac{1}{p_1Z_{01}} & \frac{1}{p_1Z_{01}} & 0 & 0 \\ 0 & 0 & \frac{1}{p_2Z_{02}} & -\frac{1}{p_2Z_{02}} \end{bmatrix} \tag{B9}$$

resulting into (38), (39), (40) and (41).